Quantum Field Theory Homework 4

Due 11 June 2021

1 Baby generating functional

Define the "Baby" path integral, with interaction term and source, as

$$Z(J) \equiv \int \frac{d\phi}{\sqrt{2\pi}} \exp\left[J\phi - \frac{\phi^2}{2} - \frac{\lambda\phi^4}{24} \right] . \tag{1}$$

This is called the "generating functional for n-point functions." Show that

$$\frac{d^n}{dJ^n} Z(J) = \int \frac{d\phi}{\sqrt{2\pi}} \phi^n \exp\left[J\phi - \frac{\phi^2}{2} - \frac{\lambda\phi^4}{24} \right] \equiv \langle \phi^n \rangle_J \, . \tag{2}$$

Go through the formal steps to demonstrate that

$$Z(J) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\lambda d^4}{24(dJ)^4} \right)^n \exp\left[\frac{J^2}{2} \right] \right) \Bigg|_{J=0} . \tag{3}$$

Now perform a double expansion of $Z(J)$ in J and λ through order J^4 and λ^2 (not necessarily using Eq. (3)–use whatever approach you find most efficient). That is, evaluate the J^0, J^2, and J^4 terms in the J expansion through $O(\lambda^2)$. Express your answer as a result for $Z(0)$, $\langle \phi^2 \rangle_{J=0}$, and $\langle \phi^4 \rangle_{J=0}$ through $O(\lambda^2)$.

Divide your results by $Z(0)$ to find the vacuum-bubble-removed correlation functions, again as formal series through order λ^2.

Define

$$W(J) \equiv \ln Z(J) \, , \tag{4}$$

and call its variations

$$\frac{d^n}{dJ^n} W(J) \Bigg|_{J=0} \equiv \langle \phi^n \rangle_{\text{conn}} \, , \tag{5}$$

the connected correlation functions.

Take the logarithm of the formal series in J and λ you found above, and expand it as a formal series in J and λ, again to order J^4 and λ^2. Read off the connected 2-point and 4-point functions to $O(\lambda^2)$.

Now draw all diagrams without vacuum bubbles for the two-point and four-point functions, through order λ^2. Evaluate them and show that they give the same answer for the correlation functions and the connected correlation functions you just found. (Note: the Feynman rules are simpler since spacetime has only one point: $\int dz = 1$ and $\Delta(x-y) = 1$. Also $-i\lambda$ is instead $-\lambda$. Therefore the value of a Feynman diagram is simply the symmetry factor times $(-\lambda)$ factors. Crib as much of the computation as possible from previous homeworks.)

2 High-order processes

Consider the theory of one real scalar field ϕ with Lagrangian density

$$\mathcal{L}[\phi, \partial_\mu \phi] = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{24}\phi^4 \,. \tag{6}$$

In class we considered the scattering of two scalars ϕ into two scalars ϕ, which involved investigating the four-point function $\langle 0| \mathbf{T}\Big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Big) |0\rangle$. It is also possible for the final state to contain more than two particles. The simplest case is that it contains 4 final state particles; this requires evaluating the 6-point correlation function

$$\langle 0| \mathbf{T}\Big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x_5)\phi(x_6)\Big) |0\rangle .$$

Draw the simplest connected Feynman diagrams contributing to this correlation function which you can find. Now many occurrences of λ are involved? Therefore, how many powers of λ will occur in the *rate* for this process (remembering that the rate involves the square of the matrix element $|\mathcal{M}|^2$)? If λ is small, do you expect that the rate for this process will be smaller or larger than the rate for the scattering process with two final state ϕ particles (assuming both are energetically allowed)?

3 M and N

In class we saw that the commutator of two Lorentz symmetry generators is given by

$$\Big[J^{\mu\nu} \,,\, J^{\alpha\beta}\Big] = i\left(\eta^{\nu\alpha}J^{\mu\beta} + \eta^{\mu\alpha}J^{\beta\nu} - \eta^{\mu\beta}J^{\alpha\nu} - g^{\nu\beta}J^{\mu\alpha}\right) . \tag{7}$$

We also introduced the (3D notation) rotation and boost generators

$$J_i = \frac{1}{2}\epsilon_{ijk}J_{ij} \,, \quad K_i = J_{i0} \,. \tag{8}$$

In each expression, the two-index object is written in terms of 4D covariant indices, while the one-index object is written in terms of 3D indices with positive metric.

First, show that

$$\exp\left(\frac{-i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = \exp\left(-i\theta_i J_i - ib_i K_i\right) \tag{9}$$

for the usual meaning of the rotation angles θ_i and boost magnitudes b_i.

Next, show that the commutation relations for the $J^{\mu\nu}$ really do lead to the commutation relations

$$\begin{aligned} \Big[J_i \,,\, J_j\Big] &= i\epsilon_{ijk}J_k \,, & (10)\\ \Big[K_i \,,\, J_j\Big] &= i\epsilon_{ijk}K_k \,, & (11)\\ \Big[K_i \,,\, K_j\Big] &= -i\epsilon_{ijk}J_k \,. & (12) \end{aligned}$$

Next, introduce the definitions

$$M_i \equiv \frac{J_i + iK_i}{2}\,, \qquad N_i \equiv \frac{J_i - iK_i}{2}\,, \tag{13}$$

and use the above commutation relations for J and K to show that

$$\Big[M_i\,,\,M_j\Big] = i\epsilon_{ijk}M_k\,, \tag{14}$$

$$\Big[N_i\,,\,N_j\Big] = i\epsilon_{ijk}N_k\,, \tag{15}$$

$$\Big[M_i\,,\,N_j\Big] = 0\,. \tag{16}$$

4 Lorentz algebra representations

Show that the Clifford algebra

$$\Big\{\gamma^\mu\,,\,\gamma^\nu\Big\} = 2g^{\mu\nu}\mathbf{1} \tag{17}$$

and the definition

$$\mathcal{S}^{\mu\nu} = \frac{i}{4}\Big[\gamma^\mu\,,\,\gamma^\nu\Big] \tag{18}$$

are sufficient to prove that

$$\Big[\mathcal{S}^{\mu\nu}\,,\,\gamma^\alpha\Big] = i(\eta^{\nu\alpha}\gamma^\mu - \eta^{\mu\alpha}\gamma^\nu) \tag{19}$$

and use this result to show that $\mathcal{S}^{\mu\nu}$ satisfies the same commutation relations as the generators of the Lorentz algebra $M^{\mu\nu}$. The representation they generate is called the *spinor representation*.

For extra credit, show that Eq. (19) (which is the transformation rule for a vector) and the definition $S(\omega) = \exp(-i\omega_{\mu\nu}\mathcal{S}^{\mu\nu}/2)$ guarantees $S^{-1}(\omega)\gamma^\mu S(\omega) = \gamma^\mu + \omega^\mu{}_\nu\gamma^\nu$ for infinitesimal ω, as claimed in class.