Quantum Field Theory Homework 5

Due 25 June 2021

1 Euler-Lagrange and Dirac equations

Consider the Lagrangian

$$\mathcal{L} = \bar{\psi}(-m + i\gamma^\mu \partial_\mu)\psi \tag{1}$$

where the ∂_μ acts to its right, that is, on ψ.

Since $\bar{\psi}$ is proportional to the conjugate of ψ, one may treat it as an independent field. [That is, considering ψ and $\bar{\psi}$ as independent fields and making variations with respect to each is equivalent to treating the real and imaginary parts of ψ as independent and $\bar{\psi}$ as composed of these real and imaginary parts. If you don't buy this, you should verify that the results you get for the first two derivations are unchanged if you treat the real and imaginary parts as the independent variables.] Find the Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \tag{2}$$

which is called the Dirac equation. What was a little funny about the derivation?

Now find the Euler Lagrange equation obtained by varying with respect to ψ,

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \tag{3}$$

which should be a relation on $\bar{\psi}$. Using

$$\gamma_\mu^\dagger \gamma^0 = \gamma^0 \gamma_\mu \tag{4}$$

and that γ^0 is nonsingular, show that the dagger of the Euler-Lagrange equation you find is equivalent to the Dirac equation.

Now show that the original Lagrangian differs from the following Lagrangian,

$$\mathcal{L}' = -m\bar{\psi}\psi - i(\partial_\mu \bar{\psi})\gamma^\mu \psi \,, \tag{5}$$

by a total derivative; so it should give equivalent physics. Derive the Euler-Lagrange equation for ψ from this Lagrangian, which should again return the Dirac equation.

2 Gamma matrix identities

Using only that the γ matrices are 4×4 matrices satisfying the Clifford algebra, and using the definition

$$\not{a} \equiv \gamma^\mu a_\mu \,,$$

verify the following:

$$\begin{aligned}
\not{k}\not{k} &= k^2 && (6)\\
\not{k}\not{p}\not{k} &= 2p\cdot k\not{k} - k^2\not{p} && (7)\\
\gamma^\mu\gamma_\mu &= 4 && (8)\\
\gamma^\mu\not{k}\gamma_\mu &= -2\not{k} && (9)\\
\gamma^\mu\not{p}\not{k}\gamma_\mu &= 4p\cdot k && (10)\\
\gamma^\mu\not{p}\not{k}\not{q}\gamma_\mu &= -2\not{q}\not{k}\not{p}\,. && (11)
\end{aligned}$$

Hint: DO NOT multiply any 4×4 matrices to do this problem! Just use repeatedly that $AB = \Big\{A\,,\,B\Big\} - BA$ and recycle each identity as you prove successive ones.

3 Gamma-5

Define $\gamma_5 \equiv \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. We will find some of its properties. In this problem we will avoid using the explicit expressions for the gamma matrices but will rely instead on their basic properties.

Using only the Clifford algebra

$$\Big\{\gamma^\mu\,,\,\gamma^\nu\Big\} = 2\eta^{\mu\nu}\mathbf{1} \tag{12}$$

show that

$$\left(\gamma^5\right)^2 = \mathbf{1}\,, \qquad \Big\{\gamma^5\,,\,\gamma^\mu\Big\} = 0 \quad \forall\mu\in\{0,1,2,3\} \tag{13}$$

(where $\mathbf{1}$ is the 4×4 identity or unit matrix).

Use the definition of γ^5, the periodicity of the trace, and the Clifford algebra to show that

$$\mathrm{Tr}\left(\gamma^5\right) = \mathrm{Tr}\left(-\gamma^5\right) \quad \text{and therefore} \quad \mathrm{Tr}\left(\gamma^5\right) = 0 \tag{14}$$

and combine this with $(\gamma^5)^2 = \mathbf{1}$ to show that γ^5 has two eigenvalues equal to 1 and two eigenvalues equal to -1.

Next, use the Clifford algebra and the fact that $(\gamma^0)^\dagger = \gamma^0$ while each $(\gamma^i)^\dagger = -\gamma^i$ to show that

$$\left(\gamma^5\right)^\dagger = \gamma^5 \tag{15}$$

is Hermitian. Hint: take the Hermitian conjugate of the explicit expression for γ^5. Use the hermiticity properties to express the result in terms of gamma matrices (without daggers), and use the Clifford algebra repeatedly to re-organize the order of the operators to get it back to the original form.

Next show that

$$P_L \equiv \frac{\mathbf{1}-\gamma^5}{2}\,, \qquad \mathrm{Pr} \equiv \frac{\mathbf{1}+\gamma^5}{2} \tag{16}$$

form a complete basis of projection matrices, in the sense that

$$P_L^2 = P_L\,, \quad \overset{2}{\mathrm{Pr}} = \mathrm{Pr}\,, \quad P_L\,\mathrm{Pr} = 0\,, \quad P_L + \mathrm{Pr} = \mathbf{1} \tag{17}$$

Combine with what we know about the eigenvalues of γ^5 to argue that each projector has two $+1$ eigenvalues and two 0 eigenvalues.

Finally, show that $\mathcal{S}^{\mu\nu}$ defined in the last problem commutes with γ^5, that is,

$$\left[\mathcal{S}^{\mu\nu}\,,\,\gamma^5\right] = 0 \quad \forall\,\{\mu,\nu\}\,. \tag{18}$$

That means that we can seek simultaneous eigenoperators of P_L, Pr and representations of $\mathcal{S}^{\mu\nu}$. Specifically (don't show this), the upper two (left-handed) components of the Dirac spinor are preserved and the lower two set zero by P_L, and vice versa for Pr.

4 Extra credit: Grassmann integrals

For those who want to know more about Grassmann integration, here is an extra problem for you. If you want to just assume that "someone knows what they are doing, I will just trust it," you can just skip this problem.

We have introduced Grassmann or anticommuting numbers. In this problem I will use the symbol Θ to write all Grassmann numbers; when there is more than one of them, I will index them Θ_1, Θ_2.

To define the path integral, I have to define integration. We require integration to obey the following three rules, in analogy to how normal integration works:

$$\begin{aligned}
\frac{d}{d\Theta}\int d\Theta A(\Theta) &= 0 \qquad \text{Integrating removes } \Theta\text{-dependence} \\
\int d\Theta\,\frac{d}{d\Theta}A(\Theta) &= 0 \qquad \text{Integration by parts} \\
\int d\Theta_1\; A(\Theta_1)B(\Theta_2) &= \left[\int d\Theta_1 A(\Theta_1)\right]B(\Theta_2)\,.
\end{aligned}$$

The last equation shows that if the integrand is the product of something which DOES depend on Θ_1 and something which does not, then the part which does not can be factored out of the integral. BUT CAREFUL: the order that we factor things out matters.

Integration also obeys the usual "distribution" rule:

$$\int d\Theta(c_1 A(\Theta) + c_2 B(\Theta)) = c_1 \int d\Theta\ A(\Theta) + c_2 \int d\Theta\ B(\Theta)$$

where c_1, c_2 are ordinary numbers – that is, the integral of a linear combination is a linear combination of the integrals, as usual.

A) Show as a consequence of the first relation that

$$\int d\Theta\ \Theta = \text{const}$$

that is, you get a constant without any further factors of Θ.

B) Show as a consequence of the second relation that

$$\int d\Theta\ 1 = 0 \qquad \text{Hint: } 1 = \frac{d}{d\Theta}\Theta\,.$$

By convention we choose the constant above to be 1, that is,

$$\int d\Theta\ \Theta = 1.$$

C) Show using this normalization that

$$\int d\Theta_1 \int d\Theta_2\ \Theta_2\Theta_1 = 1$$

BUT that

$$\int d\Theta_2 \int d\Theta_1\ \Theta_2\Theta_1 = -1$$

which means that *the order of integrations can be exchanged, but again, there are minus signs; two integration symbols anticommute.*

Now for the fun part of the problem. For this part, I imagine that the Grassmann numbers are split into two subsets, the $\bar{\Theta}$ and the Θ, just as the fermionic fields are $\bar{\psi}$ and ψ. So $\bar{\Theta}_1, \Theta_1, \bar{\Theta}_2, \Theta_2$ are four independent Grassmann numbers.

Exponentiation can be defined based on the Taylor series for the exponential function.

D) Show that exponentials are super simple:

$$\begin{aligned}
\exp(m\bar{\Theta}\Theta) &= 1 + m\bar{\Theta}\Theta \\
\exp(m_1\bar{\Theta}_1\Theta_1 + m_2\bar{\Theta}_2\Theta_2) &= 1 + m_1\bar{\Theta}_1\Theta_1 + m_2\bar{\Theta}_2\Theta_2 + m_1 m_2 \bar{\Theta}_1\Theta_1\bar{\Theta}_2\Theta_2
\end{aligned}$$

that is, show that all higher-order terms in the Taylor series are automatically zero! (Here m, m_1, m_2 are ordinary numbers.)

E) Integrate over exponentials! Use what you learned about exponentials and about integrating to show that:

$$\int d\Theta d\bar{\Theta}\ \exp(m\bar{\Theta}\Theta) \;=\; m$$

$$\int d\Theta_1 d\bar{\Theta}_1 d\Theta_2 d\bar{\Theta}_2\ \exp\left(\sum_{i,j=1,2} m_{ij}\bar{\Theta}_i\Theta_j\right) \;=\; m_{11}m_{22} - m_{12}m_{21}$$

We see a pattern here: the integral gave the determinant of the matrix m_{ij}. This holds for any number of indices, but it would be a little too much work to show it in a homework.

F) Finally, show that for $\bar{\Theta}, \Theta, \bar{\eta}, \eta$ four Grassmann numbers, that

$$\int d\Theta d\bar{\Theta}\ \exp\left(m\bar{\Theta}\Theta + \bar{\eta}\Theta + \bar{\Theta}\eta\right) = m \exp\left(\pm m^{-1}\bar{\eta}\eta\right)$$

and determine whether the $\pm$ is a $+$ or a $-$. Note that the *inverse* of the "matrix" m appears here; we could consider the $N \times N$ case with a matrix m_{ij} and we would find that the inverse matrix appears in the integration, but I won't ask you to do that in this homework.