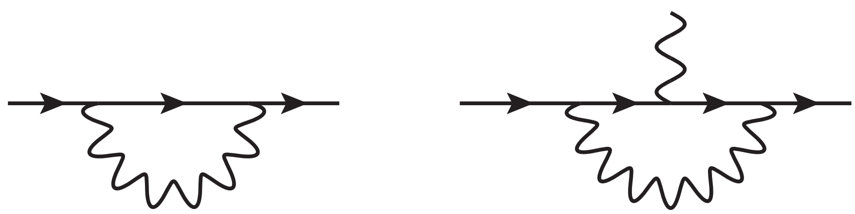
Quantum Field Theory Homework 7

Due 23 July 2021

This homework will study a few models at the loop level. Some problems will require more detailed calculations than others. But generally this homework will be harder than those which came before it. Do what you can! Ask for help if you need!

1 Gauge fixing dependence again

Consider two diagrams in QED: the one loop self-energy correction to the propagator, and the one loop self-energy correction to the vertex:



Recall that the propagator is of form

$$G^{\mu\nu}(q) = \frac{-i}{q^2 + i\epsilon}\left(g^{\mu\nu} + (\alpha - 1)\frac{q^\mu q^\nu}{q^2}\right). \tag{1}$$

Write an expression for each diagram. Separate off ONLY the part linear in α and consider it separately.

First look at the vertex correction (the right diagram). Show that the linear-in-α contribution is logarithmically divergent in 4 spacetime dimensions. Write the term with the highest possible power of q and show that it is proportional to γ^μ, that is, that the gamma-matrix structure is the same as the uncorrected vertex.

Now look at the self-energy correction (left diagram). Consider an incoming momentum p. Show that the correction *appears* to be linearly divergent, but after we combine denominators and shift integration variables to make the denominators as simple as possible, the linear divergence turns out to be odd in an integration variable and vanishes on angle averaging. Show that the remaining logarithmic divergence is proportional to $\not{p}$.

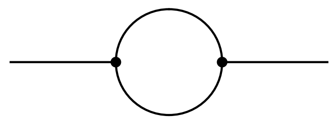
It turns out that these divergences are really there – α dependence! But they correspond to the *same* renormalization of the ψ field and can be removed by a multiplicative shift in that field. For really extra extra credit, show this in detail.

2 ϕ^3 theory

Consider the theory with one real scalar field ϕ and Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{6}\phi^3 \,. \tag{2}$$

First consider the two-point function, and the "interesting" diagram which corrects the two-point function:

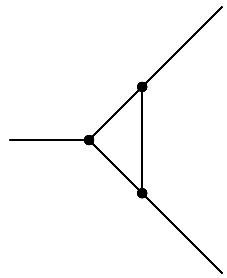


Show that this diagram is logarithmically divergent in 4 dimensions (for instance, just by counting powers of the loop momentum ℓ). Show that the divergence is independent of the external momentum, that is, if a momentum p is entering the loop, then there is no divergent term of form $p^2 \times \int d^4l/l^4$.

Then compute the value of this diagram in dimensional regularization with external momentum p, dimension $D = 4 - 2\epsilon$, and renormalization scale μ, expanding in small ϵ. Do this by combining denominators with the Feynman trick, shifting integration variables so the denominator involves $(\ell^2 - m^2 - x(1-x)q^2 + i\epsilon)^2$, and Wick rotating the ℓ^0 integration. Attempt to compute the resulting integral in dimensional regularization. You should find a $1/\epsilon$ term which is p-independent, and an ϵ^0 term which does depend on p – you may not be able to evaluate the x-integral, but at least try to find the limiting behavior for $p^2 \ll m^2$ and for $p^2 \gg m^2$. This will take some work. Get as far as you can.

For extra credit, imagine changing the action to $\mathcal{L} - (\delta m^2/2)\phi^2$ and treating the new δm^2 term as also a perturbation. Draw the one two-point diagram linear in δm^2 (it's super simple). Propose a value of δm^2 such that this diagram will cancel the contribution you found for the one-loop diagram *if* $p^2 = m^2$. Can you make the two cancel at *every* value of p^2? [This is mass renormalization.]

Next consider the $\langle\phi\phi\phi\rangle$ correlator. At lowest order there is one connected diagram, and at the next order there is one "interesting" connected diagram (cannot be interpreted as a propagator correction):



Evaluate this diagram WHEN every incoming momentum is exactly zero. You should find that the answer is simply finite in 4 dimensions, implying that the coupling g does *not* renormalize. If you cannot actually carry out the integrals, at least name the loop momentum ℓ and give the ℓ-counting argument which shows that the diagram is finite and does not diverge at large ℓ.

In what spacetime dimension does this diagram first become divergent? Hint: either look at ℓ counting, or evaluate in D dimensions and look at when your expression first encounters a pole.

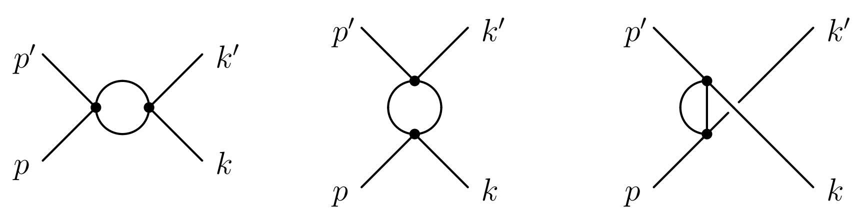
3 Simple 1-loop calculation

Consider the theory of one real scalar with a $\lambda\phi^4$ interaction:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - \frac{\lambda}{24}\phi^4 \,. \tag{3}$$

Suppose we want to compute the scattering amplitude for the process in which two particles of momenta p, p' scatter to two particles of momentum k, k'.

At leading order there is one diagram which gives $\mathcal{M} = \lambda$, as we have seen. At NLO there are three diagrams which are not simply self-energy corrections:



Name the three $\mathcal{M}$ contributions $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$, so that at NLO, $\mathcal{M} = \lambda + \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$. Show that the contribution $\mathcal{M}_2$ is

$$\mathcal{M}_2 = i\frac{(-i\lambda)^2}{2}\int \frac{d^4l}{(2\pi)^4}\frac{i}{l^2 - m^2 + i\epsilon}\,\frac{i}{(l+q)^2 - m^2 + i\epsilon} \tag{4}$$

where $q = p - k$. Write similar expressions for $\mathcal{M}_1$, $\mathcal{M}_3$; they should differ only in the definitions of q. In the end we will only need q^2 for each diagram; write each q^2 in terms of the Mandelstam variables.

Use the Feynman parameter trick to write the integrand of $\mathcal{M}_2$ in terms of a single fraction. Shift the l integration variable so that the answer can be expressed as

$$\mathcal{M}_2 = \frac{i\lambda^2}{2}\int_0^1 dx \int \frac{d^4 l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^2} \tag{5}$$

where Δ is a function of x, m, and q^2. What is the explicit form of Δ?

Next, Wick rotate the l^0 integration to obtain

$$\mathcal{M}_2 = -\frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4}\frac{1}{(l_E^2+\Delta)^2}\,, \tag{6}$$

where l_E^2 is the positive sum of the squares of the four components, $l_E^2 = \vec{l}^2 + l_{0E}^2$. Verify the overall sign.

Perform the l_E integral using *cutoff regularization*, which performs the integral over l_E from 0 to Λ, a UV cutoff. (In this scalar field theory we can get away with this.) Keep all logarithms of Λ, but drop any inverse powers of Λ. Your result should involve $\ln(\Lambda^2/\Delta)$.

Define

$$\begin{aligned} F(A) &= \int_0^1 dx \ln\Big(1 + x(1-x)A\Big) &(7)\\ &= -2 + \sqrt{\frac{4+A}{A}}\left[\log\left(1+\sqrt{\frac{4+A}{A}}\right) - \log\left(-1+\sqrt{\frac{4+A}{A}}\right)\right]\,, &(8)\end{aligned}$$

and express the result for the x integration in terms of this function. In terms of $F(A)$, express the complete result for the NLO matrix element.

For extra points, argue that your result for $\mathcal{M}_1$ has an imaginary part, and find an expression for that imaginary part – in the limit of small mass if you find that approximation necessary. For extra extra credit, explain what this imaginary part has to do with the optical theorem.