Path Integral

So far: $\mathcal{L}(\varphi_a, \partial_\mu \varphi_a) = \frac{1}{2}\partial_\mu \varphi_a \partial_\mu \varphi_b M_{ab} - V(\varphi_a)$

We want cross-sections

$$\sigma(p_1, p_2 \to k_1 \dots k_n) = \frac{1}{2E_1 2E_2 |v_1 - v_2|} \int_{\text{range}} \frac{d^3k_1 \dots d^3k_n}{(2\pi)^3 2E_{k_1} \cdots (2\pi)^3 2E_{k_n}} (2\pi)^4 \delta^4(p_1 + p_2 - \Sigma k_n) \, |M|^2$$

$$\langle 0 | T(\varphi(x_1) \dots \varphi(x_n)) | 0 \rangle = G(x_1, \dots x_n)$$

Our Goal

$$G(p_1, \dots, p_n) = \left(\frac{iZ}{p_1^2 - m^2 + i\epsilon}\right)\left(\frac{iZ}{p_2^2 - m^2 + i\epsilon}\right) \dots (2\pi)^4 \delta^4(p_1 + \dots + p_n)\, iM$$

2 Approaches: Assume $V(\varphi) = \frac{M_{ab}}{2}\varphi_a\varphi_b + \frac{\lambda_{abcd}}{24}\varphi_a \cdots \varphi_d$

small.

Solve thy w.o. λ $\rightarrow$

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k}\left(e^{-ik\cdot x} a_k + e^{+ik\cdot x} a_k^\dagger\right)$$

$\rightarrow$ Pert. Thy.

If $\lambda \neq 0$, is not true etc.

m of part may not = mass

$|0\rangle_{\text{free thy}} \neq |0\rangle_{\text{int.}}$ $\langle 0|0\rangle_{\text{free, int.}} = 0$

Alternately - 1) Build general approach: Path Int.

2) Learn how to use it if λ small

$\rightarrow$ Pert. Thy.

$$\mathcal{L} \longrightarrow H = \frac{1}{2}\pi_a M^{-1}_{ab}\pi_b + V(\varphi_a)$$

Time evol'n : e^{-iHt} But First

Modify $\mathcal{L} \longrightarrow \partial_\mu\varphi_a \partial_\mu\varphi_b \frac{M_{ab}}{2} - V(\varphi) - J_a(x)\varphi_a(x)$

J_a : "external" sp. dep. funct. which I can choose

I want thy with $J_a = 0$.

$$H = \int \left(\frac{1}{2}\pi_a M^{-1}_{ab}\pi_b + V(\varphi_a) + J_a(x)\varphi_a(x) + \frac{1}{2}\vec{\nabla}\varphi_a M_{ab}\nabla\varphi_b \right) d^3\vec{x}$$

Int. picture: $\underbrace{\int d^4x \, J(x_\mu)\varphi(x_\mu)}_{} = H_I$

$H_{\text{true original}} = H_0$

$$\langle 0|_{\text{future}} \; T\exp\left[-i\int_{t_0}^{t_f} dt\, H_I\right] |0\rangle_{\text{past}} = Z(J)$$

$$\langle 0| (1 - iH_I(t_f)\Delta t)(1 - iH_I(t_f - \Delta t)\Delta t) \cdots (1 - iH_I(t_0)\Delta t) |0\rangle$$

$$\frac{\delta Z(J)}{\delta J(y)} = \langle 0| (\ )(\ )(\ ) - i\varphi(y)(\ )(\ )(\ )(\ ) |0\rangle$$

$$\left.\frac{\delta^n Z(J)}{\delta J(y_1)\cdots\delta J(y_n)}\right|_{J=0} = \langle 0|(\ )(-i\varphi(y_n))(\ ) - i\varphi(y_{n-1}) \cdots |0\rangle$$

$$= \langle 0| T(\varphi(y_n)\varphi(y_{n-1}) \cdots \varphi(y_1))|0\rangle \Big|_{J=0}$$

Moral:

$Z(\underline{J})$ Generating functional for all T-ordered correl. funcs of thy

$$\boxed{Z(J) = \langle 0| T\exp\left(-i\int H_I\, dt'\right)|0\rangle}$$

$$\left.\frac{\delta^n Z}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} = (-i)^n \langle 0| \underline{T}\left(\varphi(x_1)\cdots\varphi(x_n)\right)|0\rangle$$

$$'' \quad G_n(x_1, \dots x_n)$$

$$\underline{Z(J)} = 1 + \int_x \underline{J(x)}\, \underline{G_1(x)} + \int_{xy} \frac{J(x)J(y)}{2} \underline{G_2(x,y)}$$

$$+ \int_{xyz} \frac{J(x)J(y)J(z)}{6} \underline{G_3(x,y,z)} + \cdots$$

So, how do I compute $Z(J)$???

Go to Quantum Mechanics of n generalized coord

$Q_a \quad a = 1, \ldots, n$

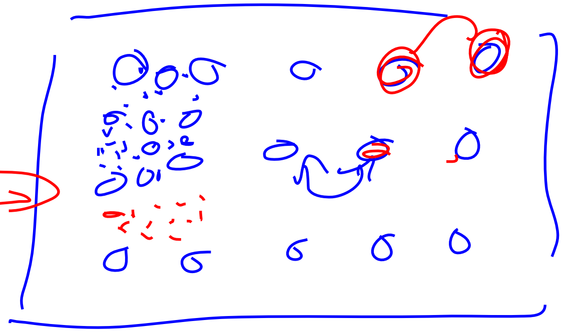
$P_a \quad a = 1, \ldots, n$

$$H = \frac{1}{2} P_a M^{-1}_{ab} P_b + V(Q)$$

incl. $\left(\frac{Q_a - Q_b}{\text{spac}}\right)^2 \sim (\nabla Q)^2$

includes $J_a Q_a$ and other stuff

To get to QFT,



Q_a are φ at each latt pt.

(10^{156} Gen. coord)

"regulation" present from start!!

$$Z_{fi}(J) = \langle \psi_f \,|\, U(t_f - t_i) \,|\, \psi_i \rangle$$

fin-state   time evol   in. state

Break time evol into many (N) small $(\Delta t/N)$ steps

$\Delta t = t_f - t_i$. $\lim_{N\to\infty}$ implizit $t_f = t_N$ $t_i = t_0$

$$Z_{fi} = \langle \psi_f | \underbrace{\left(U(t_N - t_{N-1})\, U(t_{N-1}, t_{N-2}) \cdots\cdots U(t_1 - t_0)\right)}_{N \text{ op's.}} | \psi_i \rangle$$

Each op, I need to include $\mathcal{O}(1/N)$ effects. Not $1/N^2$ eff.

$$\exp -i\int_{t_{n-1}}^{t_n} H\,dt \simeq 1 - iH\underbrace{\frac{\Delta t}{N}}_{\text{need}} - \frac{H^2}{2}\left(\frac{\Delta t}{N}\right)^2$$

N^{-2} don't need.

$$\int \frac{dP_1 \cdots dP_m}{(2\pi)^m} |P\rangle\langle P| = \mathbb{1}$$

$$H = \underbrace{P_a M^{-1}_{ab} P_b}_{2} + V(Q)$$

easy on P-basis. easy to eval. on coord Basis

Recall $\underbrace{\int dQ_1 \cdots dQ_m |Q\rangle\langle Q|}$ $= \mathbb{1}$

Insert N^+ copies of $\int dQ_a(t_n) |Q(t_n)\rangle\langle Q(t_n)|$

n int's

$$Z_{fi} = \int dQ_a(t_0)\, dQ_a(t_1) \cdots dQ_a(t_N) \times$$

$(N+1) * (n)$ int's

$$\langle \psi_f | Q(t_N)\rangle \langle Q(t_N) | U(t_N, t_{N-1}) | Q(t_{N-1})\rangle \langle Q(t_{N-1}) | U(t_{N-1}, t_{N-2}) | Q(t_{N-2})\rangle$$

what is this

$$\cdots = \langle Q(t_0) | \psi_i \rangle$$

This will be

$$\int \mathcal{D} Q_a(t) \quad \text{Path Integral}$$

$$\exp i \int_{t_{N-2}}^{t_{N-1}} L(Q, \dot{Q})\, dt$$

$$\langle Q(t_m)|\,\hat{\mathcal{U}}\left(\frac{\Delta t}{N}\right)|Q(t_{m-1})\rangle \qquad \hat{\mathcal{U}}\left(\frac{\Delta t}{N}\right) = 1 - i\frac{\Delta t}{N}\hat{H}$$

$$= 1 - i\frac{\Delta t}{N} M^{-1}_{ab}\hat{P}_a\hat{P}_b + i\boxed{\frac{\Delta t}{N}}V(\hat{Q})$$

$$V(\hat{Q})|Q(t_{m-1})\rangle = V(Q(t_{m-1}))|Q(t_{m-1})\rangle \text{ or } \simeq (Q(t_m)) \text{ almost same}$$

$$= \underbrace{\langle Q(t_m)|\,e^{-i\frac{\Delta t}{N}M^{-1}_{ab}P_aP_b}\,|Q(t_{m-1})\rangle} + i\frac{\Delta t}{N}V(Q_m)$$

$$|P_m\rangle\langle P_m|$$

$$\int \frac{dP_a(t_m)}{(2\pi)^{\#}}$$

$$= \int \frac{dP_1(t_m)\cdots dP_m(t_m)}{(2\pi)^m}\, e^{-i\frac{\Delta t}{N}M^{-1}_{ab}P_a(t_m)P_b(t_m)} \langle Q(t_m)|P(t_m)\rangle\langle P(t_m)|Q(t_{m-1})\rangle$$

$$\langle Q|P\rangle = e^{-iQ\cdot P}$$

$$\langle P|Q\rangle = e^{+iQ\cdot P}$$

$$\Rightarrow \int \frac{\prod dP(t_m)}{2\pi}\, e^{-i(Q(t_m))P + i(Q(t_{m-1}))P}\, e^{-i\frac{\Delta t}{N}M^{-1}PP}$$

Gaussian Integral.

what is $\prod_{a=1}^{n} \int \frac{dP_a}{2\pi} e^{\underbrace{iP_a(Q_a(t_{m-1})-Q_a(t_m)) - \frac{i\Delta t}{2N} M^{-1}_{ab} P_a P_b}} =$

$e^{\underbrace{-\frac{A}{2}P^2 - BP - \frac{B^2}{2A} + \frac{B^2}{2A}}}$

$\frac{A}{2}(P - B/A)^2 + B^2/2A$

what if A is a matrix?

$e^{\underbrace{-\frac{A_{ab}}{2}P_a P_b - B_a P_a - \frac{B_a A^{-1}_{ab} B_b}{2} + \frac{B_a A^{-1}_{ab} B_b}{2}}}$

$\frac{(P_a + A^{-1}_{ab} B_b) A_{ab} (P_b + A^{-1}_{bc} B_c)}{2}$

$A = \frac{i\Delta t}{N} M^{-1}$

$B = i(Q(t_m) - Q(t_{m-1}))$

$e^{\underline{+i \boxed{\frac{N}{\Delta t}} (Q_m - Q_{m-1}) M (Q_m - Q_{m-1})}}$

P int's $\longrightarrow$ Q-indep. coeff.: $\frac{\mathrm{Det}^{-1/2} M^{-1}}{(2\pi)^{n/2}}$ overall #

End up with $(\text{A Constant}) \times e^{-iV(Q)\frac{\Delta t}{N}} \; e^{i\frac{N}{\Delta t}(Q(t_m)-Q(t_{m-1}))M(\ldots)/2}$

$$Q(t_m)-Q(t_{m-1}) = \frac{\Delta t}{N}\left(\frac{Q(t_m)-Q(t_{m-1})}{(\Delta t/N)}\right) = \frac{\Delta t}{N}\dot{Q}$$

$$\Rightarrow e^{i\left[\frac{\Delta t}{N}\right]\underbrace{\left[\frac{\dot{Q}_a M_{ab}\dot{Q}_b}{2} - V(Q)\right]}_{L(Q,\dot{Q})}}$$

$$\langle \psi_f | Q_N \rangle \underbrace{e^{i\frac{\Delta t}{N}L}\, e^{i\frac{\Delta t}{N}L}\, e^{i\frac{\Delta t}{N}L} \cdots}_{\exp i\int dt\, L(Q,\dot{Q})} \langle Q_0 | \psi_i \rangle$$

$$\langle \psi_f | U(t_f - t_i) | \psi_i \rangle = Z_{fi}$$

$$= \int \mathcal{D}Q_a(t)\, \psi^*(\psi_f)\, \psi(\psi_i)\, e^{\frac{i}{\hbar}\int L(Q,\dot{Q})\,dt}$$

BC

Path Integral.

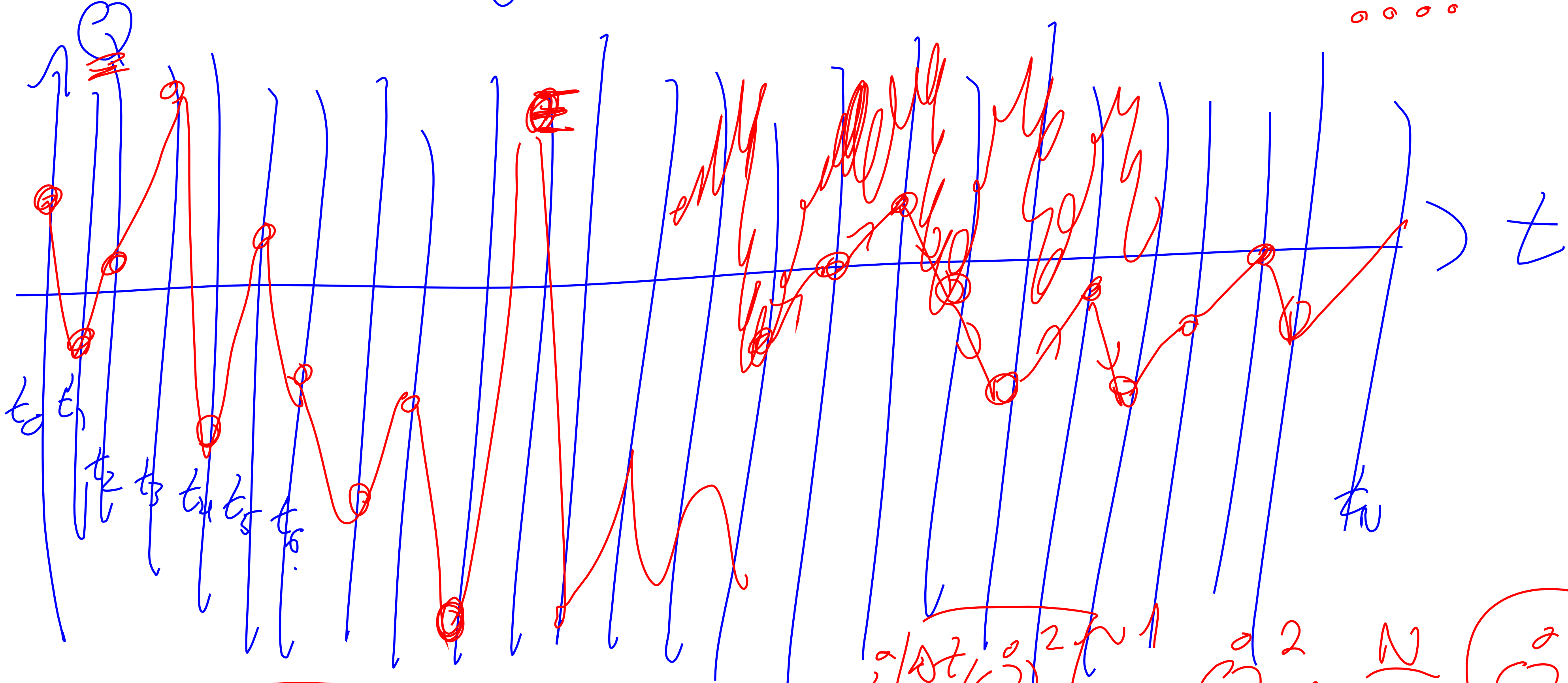
Advantage: No Operators.

Q_a are now ordinary # int variables.

Disadvantage: 1 int. per canon. coord per pt in time — $\lim_{N\to\infty}$

Math Person – Does it make sense? For finite # of coord ⟶ yes.

What is this thing? One coord



$$e^{i\frac{\Delta t}{N}(\dot Q)^2} \sim 1$$

$$\dot Q^2 \sim \frac{N}{\Delta t}$$

$$\dot Q \sim \frac{\sqrt N}{\sqrt{\Delta t}}$$

$$\int dq_0\, dq_1\, dq_2 \ldots dq_N$$

Possible history for $Q(t)$

Paths you could follow

$$\frac{\Delta t}{N}\dot Q = \Delta Q \sim \sqrt{\frac{\Delta t}{N}}$$

No huge jumps.

a^{-2}

Reformulation of QM.

Field Thy: $V(Q) \longrightarrow V(\varphi) + |\nabla\varphi|^2/2$

$$L = \int d^3x \, \mathcal{L}$$

$$Z_{fi} = \int \mathcal{D}\varphi(x^\mu) \, e^{\frac{i}{\hbar}\int dt \int d^3x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right]}$$

Lorentz Inv.

φ at each pt in sp.
at each pt in time

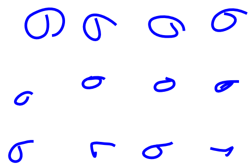
Lorentz

Lorentz

1 int per field per space-time pt

How do I use this?

1) Do the Integral. Lattice Gauge Thy



2) Approximate: λ small

Perturbation Theory (Störungstheorie)

systematic approach. Works in everything but QCD

Analytical grip