Last time: Scattering (Decay) determined by T-ordered N-PT Functions

$$G(p_1, p_2; k_1 k_2 \ldots) = \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \ldots e^{+ix_1\cdot p_1 + ix_2\cdot p_2} e^{-i(y_1\cdot k_1 + y_2\cdot k_2 + \ldots)} \langle 0|T(\phi(x_1)\ldots\phi(y_n))|0\rangle$$

$$= \left(\frac{i}{p_1^2 - m^2 + i\epsilon}\right)\left(\frac{i}{p_2^2 - m^2 + i\epsilon}\right)\cdots\left(\frac{i}{k_n^2 - m^2 + i\epsilon}\right) \times (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - \ldots - k_n)\, i\mathcal{M}$$

want

$$\langle 0|T(\phi(x_1)\ldots\phi(x_n))|0\rangle = \left(\frac{-i\delta}{\delta J(x_1)}\right)\cdots\left(\frac{-i\delta}{\delta J(x_n)}\right) Z(J)\Big|_{J=0}$$

$$Z(J) = \langle 0| U_I(t_f - t_i)|0\rangle \qquad H \to H + \int \phi J\, d^3x$$

$$Z(J) = \int \mathcal{D}\varphi(x)\, \exp\Big[+i \underbrace{\int d^4x\, \mathcal{L}[\varphi, \partial_\mu \varphi]}_{S}\Big]$$

$\varphi_{init} = \varphi_0$

$\varphi_{fin} = \varphi_0$

Norm factor

Do know $\langle 0 | 0 \rangle \Big|_{J=0} = 1$

$$Z(J=0) = 1$$

$$\frac{\partial}{\partial J} \frac{Z(J)}{Z(0)} \Bigg|_{J=0}$$

Use this to avoid computing / understanding Norm. of Path Int

$\psi_{in} = \psi_0$ is what?

$\langle 0|$ $|0\rangle$ what is this?

Free Theory vacuum $|0\rangle_0$ known.

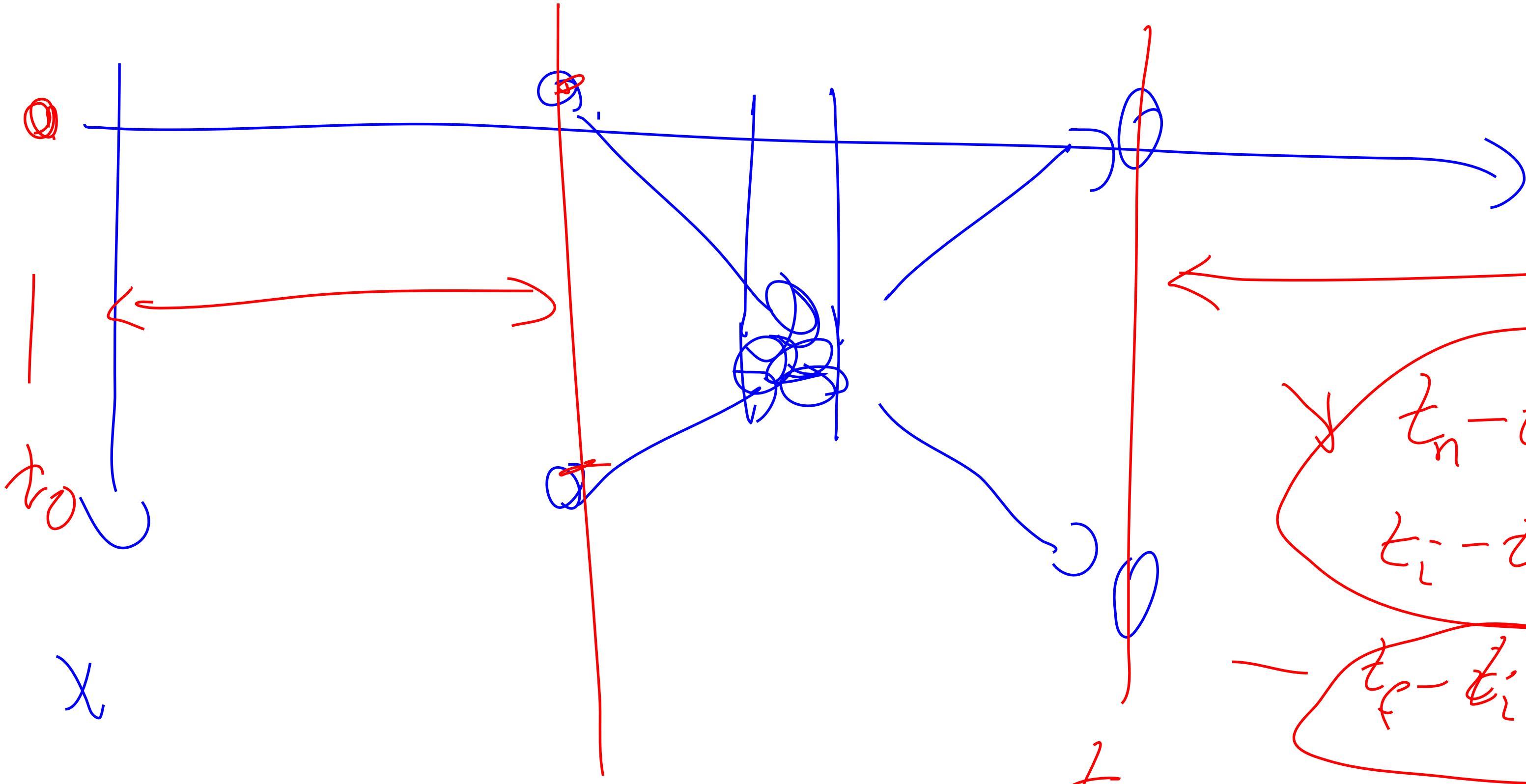
SHO vac. in each $\vec{p}$ mode.

Haag's Theorem: If $L \to \infty$ limit is taken, either $a \to 0$

$$\langle 0|0\rangle_0 = 0$$

Int. thy states have 0 overlap w. free thy states.

How to do calc if I don't know $|0\rangle$?



$|0\rangle$

t_n

$t_n - t_f$ big

$t_i - t_0$ big

$t_f - t_i$ big

$\langle 0| H t_i$

in

$\varphi\varphi$

$t_f |0\rangle$

out

$\varphi\varphi$

$\langle 0|$ ...

$|0\rangle$ ...

$|\text{stuff}\rangle = \sum_n c_n |n\rangle$

$c_0 \neq 0$ ...

$e^{-iH(t_i-t_0)}$ $e^{-\epsilon H(t_i - t_0)}$

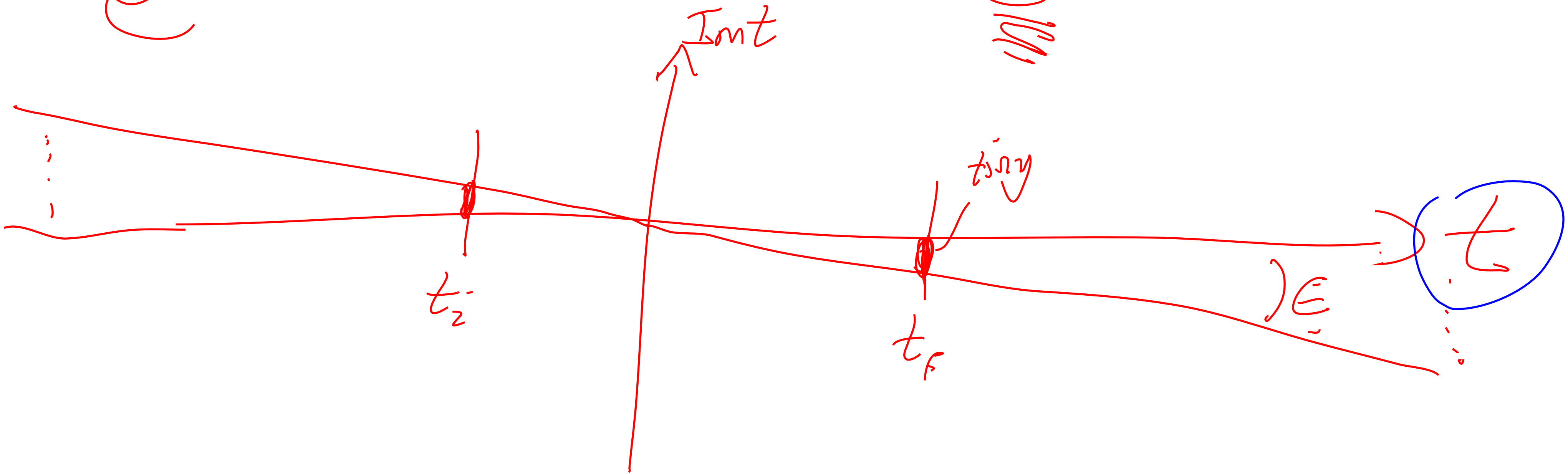
$e^{-iHt - \epsilon H(t_n - t_f)}$

$e^{-iH(t_{big}-t_{small})}$ $\quad$ $e^{-\epsilon H(t_{big}-t_{small})}$

$\epsilon \ll 1$ but $\epsilon(t_{big}-t_{small}) > \frac{1}{m}$

assume $m>0$.

$e^{(-i-\epsilon)Ht}$ $\quad$ wherever it was e^{-iHt} $\quad$ $e^{-i(1-i\epsilon)Ht}$



$$\exp((-i-\epsilon)Ht)\sum_n c_n|n\rangle =$$

$$\sum_n e^{-iE_n t}\, e^{-\epsilon E_n t}\, c_n|n\rangle$$

$$\underline{1} \qquad \text{Vacuum: } \underline{1} \qquad \underline{E_0 = 0} \qquad c_n|0\rangle$$

All other n's: $e^{-iE_n t}\, e^{-\epsilon E_n t}\, \cancel{c_n|n\rangle} \to 0$

$$\le e^{-\epsilon m t} \approx 0$$

$$c_n|0\rangle$$

Always use $e^{-i(1-i\epsilon)Ht}$

Then boundaries not important.

What does that do?

$$t \to (1-i\epsilon)t$$

$$\int dt\, f((1-i\epsilon)t)\, e^{ip^0 t}$$

$$e^{i(1+i\epsilon)p^0(1-i\epsilon)t}$$

$$p^0 \to (1+i\epsilon)p^0$$

$$\frac{1}{p_0^2 - \vec{p}^{\,2} - m^2} \sim p_0^2 - E_p^2 + i\epsilon p^0$$

$$[(1+i\epsilon)p^0]^2 \underbrace{- \vec{p}^{\,2} - m^2}$$

Calculating Path Integral.

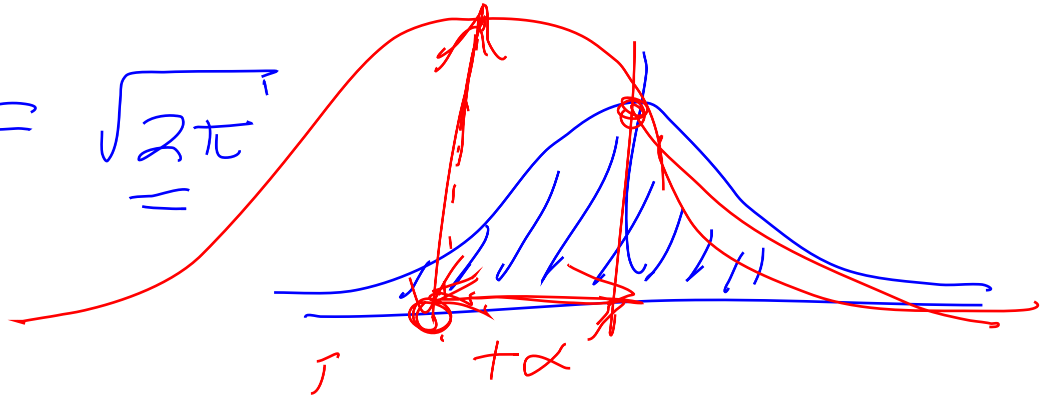
$$\int \mathcal{D}\varphi \, \exp\left(i \int d^4x \left[\frac{1}{2}(\partial_0 \varphi)^2 - \frac{(\nabla\varphi)^2}{2} - \frac{m^2}{2}\varphi^2 - \cancel{\frac{\lambda}{24}\varphi^4} - \varphi J \right] \right)$$

Easier Case: $\lambda \to 0$

$$\int \mathcal{D}\varphi \, \exp i\left[\text{something } \varphi\varphi + \text{something } \varphi \right]$$

Gaußian Integral!

Simplest: $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-x^2/2} = \sqrt{2\pi}$



$$\int_{-\infty}^{\infty} dx\, e^{-x^2/2}\, e^{\alpha x} = e^{\alpha^2/2} \int dx\, e^{-(x-\alpha)^2/2}$$

$$-\frac{x^2}{2} + \alpha x - \frac{\alpha^2}{2} + \frac{\alpha^2}{2}$$

$$e^{\underbrace{-\frac{x^2}{2} + \alpha x - \frac{\alpha^2}{2}}_{-\frac{(x-\alpha)^2}{2}} + \alpha^2/2}$$

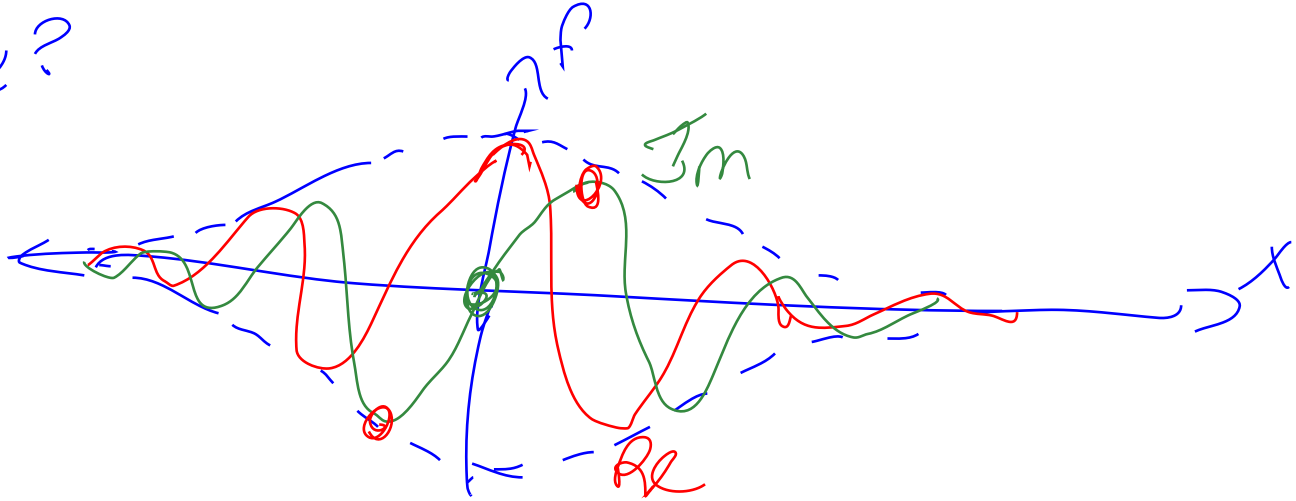
$x \to x - \alpha$

$\boxed{y = x - \alpha}$

$$e^{\alpha^2/2} \int_{-\infty}^{\infty} dy\, e^{-y^2/2} = e^{\alpha^2/2} \sqrt{2\pi}$$

What if α is complex?

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} e^{i\alpha x}$$

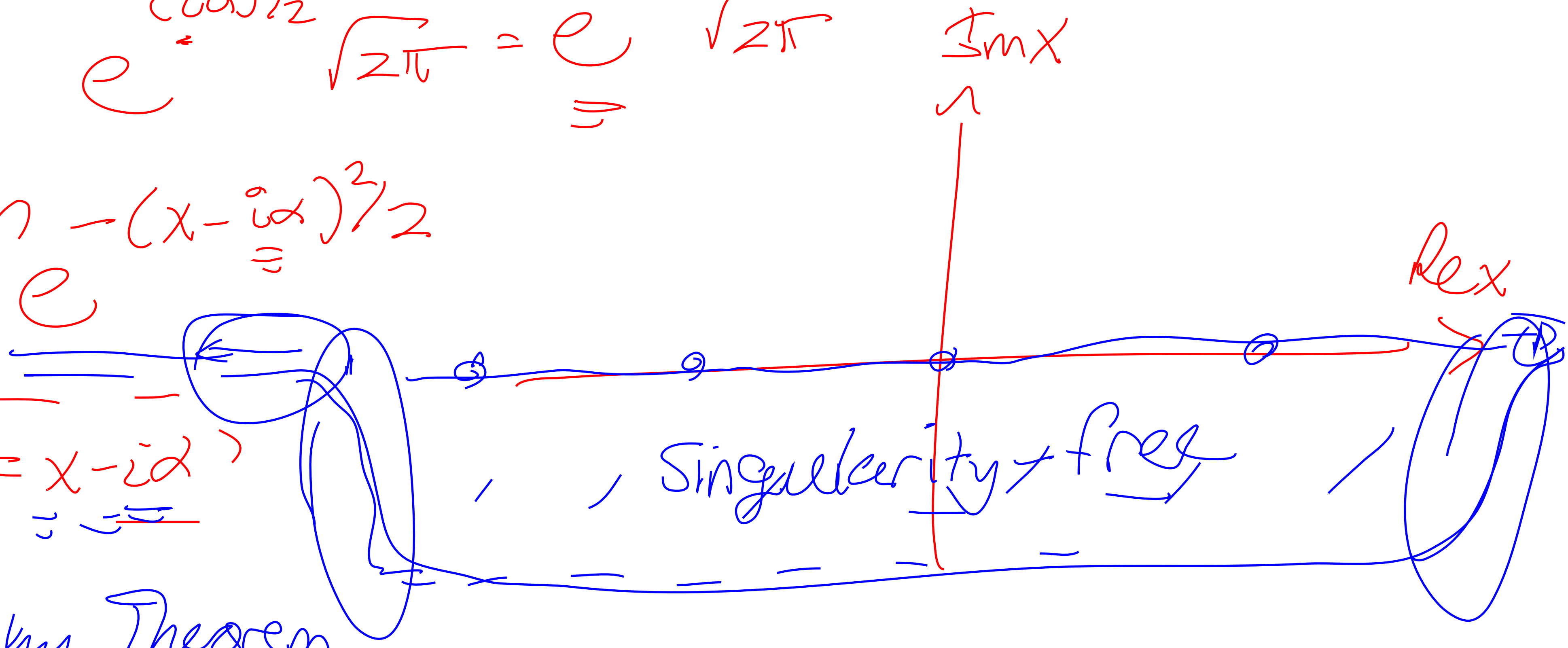


$\alpha \longrightarrow i\alpha$

$$e^{(i\alpha)^2/2}\sqrt{2\pi} = e^{-\alpha^2/2}\sqrt{2\pi}$$

$$e^{-\alpha^2/2}\int e^{-(x-i\alpha)^2/2}$$

$(y = x - i\alpha)$



Cauchy Theorem

Complex value?

$$\int e^{-\alpha x^2/2}\,dx \qquad \alpha \in \mathbb{C} \qquad e^{-iHt} \quad e^{-\epsilon H t}$$

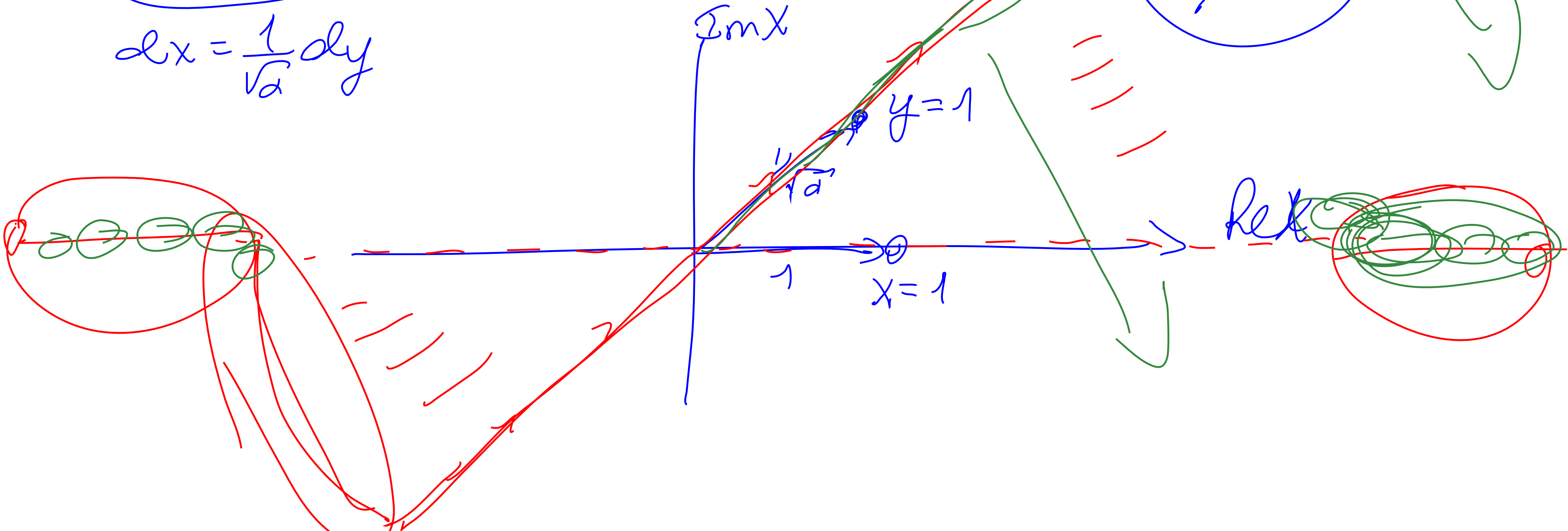
$y = \sqrt{\alpha}\, x$

$x = \frac{1}{\sqrt{\alpha}} y$

$dx = \frac{1}{\sqrt{\alpha}} dy$

$$= \int e^{-(\sqrt{\alpha} x)^2/2} \frac{d(\sqrt{\alpha}\, x)}{\sqrt{\alpha}} = \int_{-\infty}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{\alpha}} = \sqrt{2\pi/\alpha}$$

OK if $\mathrm{Re}\,\alpha \geq 0$



$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{\alpha x^2}{2} - \beta x \; - \frac{\beta^2}{2\alpha} + \frac{\beta^2}{2\alpha}} = \sqrt{\frac{2\pi}{\alpha}}\, e^{+\beta^2/2\alpha}$$

$\alpha, \beta \in \mathbb{C}$

$\underline{\mathrm{Re}\,\alpha \geq 0}$

Multi-variable? x_a $a = 1 \ldots N$ variables

$$\int_{-\infty}^{\infty} \prod_{a=1}^{N} dx_a \, \exp\left[-\sum_{a,b=1}^{N} \underbrace{M_{ab} x_a x_b} + \sum_{a=1}^{N} k_a x_a \right]$$

Generalized Quadratic Form

N int's

1) $M_{ab} = M_{ba}$ symm. If not, $M_{ab} \to \frac{1}{2}(M_{ab} + M_{ba})$

Orthogonal eigenvectors, L, R ev's same.

$$M_{ab} = \sum_i \xi_{ia} \lambda_i \xi_{ib} \qquad \xi_{ia} = \begin{bmatrix} \xi_{i1} \\ \vdots \\ \xi_{iN} \end{bmatrix}$$

Evect. λ_i eval.

$$\sum_a \xi_{ia} \xi_{i'a} = \delta_{ii'}$$

If all $\lambda_i \neq 0$, $\mathrm{Re}\,\lambda_i \geq 0$ then — in business

$M^{1/2}$, M^{-1} well defined.

Same e-vectors. E-val's: $\sqrt{\lambda_i}$, $\frac{1}{\lambda_i}$

$$\boxed{y_a = \sum_b M^{1/2}_{ab} x_b}$$

$$\underbrace{x M^{1/2}}\underbrace{M^{1/2} x} = x_a M_{ab} x_b = y_a y_a$$

$$K_a x_a = K_a M^{-1/2}_{ab} \underbrace{M^{+1/2}_{bc} x_c} = \left[K_a M^{-1/2}_{ab}\right] y_b$$

Jacobian

$$\int \prod dx_a \exp\left[-\frac{1}{2} x M x + Kx\right] = \int \prod dy_a \left[\det \frac{\partial x_a}{\partial y_b}\right] e^{-\frac{y^2}{2} - K M^{-1/2} y}$$

$$\int \prod_a dx_a \, e^{-\frac{1}{2} x_a M_{ab} x_b + K_a x_a} = \int \prod_a dy_a \, \mathrm{Det}\frac{\partial x}{\partial y} \, e^{-\frac{1}{2} y_a y_a} \, e^{+K_a M^{-1/2}_{ab} y_b}$$

$$= e^{+\frac{1}{2} K_a M^{-1}_{ab} K_b} \, (2\pi)^{N/2} \, \mathrm{Det}\left|\frac{\partial x}{\partial y}\right|$$

$$\parallel$$

$$\mathrm{Det}\, M^{-1/2} = (\mathrm{Det}\, M)^{-1/2} = \prod_i \lambda_i^{-1/2}$$

$$y_a = M^{1/2}_{ab} x_b$$

$$\frac{\partial y_a}{\partial x_b} = M^{1/2}_{ab}$$

$$\frac{\partial x_i}{\partial y_j} = M^{-1/2}_{ab}$$

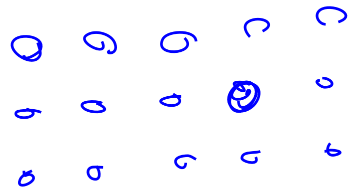
$$\prod_i \sqrt{\frac{2\pi}{\lambda_i}}$$

$$\boxed{= e^{\frac{1}{2} K_a M^{-1}_{ab} K_b} \, (2\pi)^{N/2} \, \mathrm{Det}^{-1/2} M_{ab}}$$

K, M (numbers/matrix/vector) is OK

Now consider

$$Z(J) = \int \mathcal{D}\varphi \; e^{\,i\int d^4x \left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - J\varphi\right]}$$



$$\int_x (\partial_\mu\varphi)^2 = \sum_x \left(\frac{\varphi(x+a)-\varphi(x)}{a}\right)^2$$

$$\frac{\varphi(x+a)^2 - 2\varphi(x+a)\varphi(x) + \varphi(x)^2}{a^2}$$

0 1 2 3 4 ...

$$[\varphi(0)\ \varphi(1)\ \cdots]\begin{bmatrix} +2/a^2 & -1/a^2 & & \\ -1/a^2 & +2/a^2 & -1/a^2 & \\ & -1/a^2 & 2/a^2 & \ddots \\ & & \ddots & \ddots \end{bmatrix}\begin{pmatrix}\varphi(0)\\ \varphi(1)\\ \varphi(2)\\ \varphi(3)\\ \vdots\end{pmatrix}$$

$$Z(J) = \int \mathcal{D}\varphi_a \exp\left[i \varphi_a M_{ab} \varphi_b - i \varphi_a J_a \right]$$

$$\left[\partial_\mu \partial^\mu - m^2 \right]$$

$-iM$

$$\frac{(2\pi)^{\frac{N}{2}}}{(\operatorname{Det} M)^{1/2}} \; e^{+i \frac{J_a M^{-1}_{ab} J_b}{2}}$$

$$\frac{Z(J)}{Z(0)} = e^{i \frac{J_a M^{-1}_{ab} J_b}{2}}$$

$+ K M^{-1} K$ but $K = -iJ$

$M \to -iM, \; M^{-1} \to +iM^{-1}$

$(+i)(-i)^2$

Wick's Theorem:

Consider $\displaystyle\int \mathcal{D}\varphi_a \, e^{-\frac{1}{2}\varphi_a M_{ab}\varphi_a} \, e^{+J_a\varphi_a} = Z(J)$

$$\frac{1}{Z(0)}\left[\frac{\partial^m}{\partial J_{a_1}\cdots\partial J_{a_m}} Z(J)\right]_{J=0} = \sum_{\substack{\text{all pairings}\\ \text{of } a_i}} M^{-1}_{a_{p_1}a_{p_2}} M^{-1}_{a_{p_3}a_{p_4}} \cdots M^{-1}_{a_{p_{m-1}}a_{p_m}}$$

$m=2$: $M^{-1}_{a_1a_2}$ $\quad 1$

$m=4$: $M^{-1}_{a_1a_2}M^{-1}_{a_3a_4} + M^{-1}_{a_1a_3}M^{-1}_{a_2a_4} + M^{-1}_{a_1a_4}M^{-1}_{a_2a_3}$ $\quad 1\cdot 3$

$m=6$: $M^{-1}_{a_1a_2}M^{-1}_{a_3a_4}M^{-1}_{a_5a_6} + M^{-1}_{a_1a_2}M^{-1}_{a_3a_5}M^{-1}_{a_4a_6} + \cdots$ $\quad 1\cdot 3\cdot 5$

M_{2N}: $(2N-1)(2N-3)\cdots 3\cdot 1$

New Notation

$$\underbrace{1\cdot 3\cdot 5\cdot \cdots \cdot (2N-3)(2N-1)} = \underline{(2N-1)!!}$$

Looks like $((2N-1)!)!$

$(3!)! = 6! = 720$

$3!! = 3\cdot 1 = 3$

$$= \frac{2N!}{2^N N!} = \frac{(2N)(2N-1)\cdots 3\,2\,1}{(2N)\quad (2N-2)\cdots 2}$$

\# of pairings: $\underline{2N}$ objects. How many ways to pair them?

pair a_1 with someone: $2N-1$ choices. a_m

a_1, a_m used up: $(2N-2)$ things to pair. $\longrightarrow$ $(2N-3)(2N-5)\cdots 3\cdot 1$ ways

Wick Theorem: Induction   2N-2 Deriv's. (Odd# → 0)

$$\frac{\partial^{2N-1}}{\partial J_{a_{2N}}\,\partial J_{a_{2N-1}}\cdots\partial J_{a_2}}\;\frac{\partial}{\partial J_{a_1}}\,e^{\frac{J_a M^{-1}_{ab} J_b}{2}}\Big|_{J=0}$$

$$= \underbrace{\frac{\partial}{\partial J_{a_{2N}}}\cdots}_{2N-1}\; J_a M^{-1}_{a a_1}\, e^{\frac{J M^{-1} J}{2}}$$

$$\frac{\partial}{\partial J_b} J_a x = \delta_{ab}\, x + J_a \frac{\partial}{\partial J_b} x$$

2N-1 terms

$$\delta_{a_k a_1} \quad M^{-1} \quad J \frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial} e^{\frac{J M^{-1} J}{2}}\Big|_{J=0}$$

$$\sum_{k=2}^{2N} M^{-1}_{a_k a_1}\;\underbrace{\frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}\, e^{\frac{1}{2} J M^{-1} J}}_{\text{all but } k,\,1}$$

Induction