Homework due in 1 week

Evaluations - IF you did not get link via TUCAN, tell me - I can have Assistants send it.

$$i \frac{\partial}{\partial J} Z(J) = \int \mathcal{D}\varphi \, \exp\left[i \int d^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - J(x)\varphi(x) \right) \right]$$

$\varphi\varphi\varphi$

All φ space-time histories

$\mathcal{L}$

$$= -\frac{\lambda}{24}\varphi^4(x)$$

will let us compute "everything" relevent in thy particles, masses, ...

$$\frac{\partial^2}{\partial J \partial J} Z(J) \Big|_{J=0} = G_T$$

$$\frac{\partial^{2+N}}{\partial J_1 \cdots \partial J_{N+2}} Z(J) \Big|_{J=0} \Rightarrow 2 \to N \text{ scattering}$$

$$\frac{\partial}{\partial J(x)} e^{-i\int J(y)\varphi(y)\,d^4y} = -i\varphi(x)\, e^{-i\int J\varphi \cdots}$$

$$\frac{\partial}{\partial J(x)} J(y) = \delta^4(x-y)$$

If space is discrete $x_i = a\vec{n}_i$

$x_1 \quad x_2 \quad x_3$

$$\partial_i \varphi(x) = \frac{\varphi(x+a\hat{i}) - \varphi(x)}{a} \qquad -\partial_i \varphi \partial_i \varphi = -\left[\frac{\varphi(x+a\hat{i}) - \varphi(x)}{a}\right]^2$$

$$[\varphi(x_1)\ \varphi(x_2) \cdots] \begin{bmatrix} 2/a^2 & -1/a^2 & & \\ -1/a^2 & 2/a^2 & -1/a^2 & \\ & -1/a^2 & 2/a^2 & \ddots \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \varphi(x_1) \\ \varphi(x_2) \\ \varphi(x_3) \\ \vdots \end{bmatrix}$$

x fixed y

$$\int \partial_\mu \varphi \partial^\mu \varphi \, d^4x$$

$$= \int_x \int_y \varphi(x) M_{xy} \varphi(y)$$

$$\underbrace{\frac{1}{2} \partial_\mu \partial^\mu \delta^4(x-y)}$$

$$\int d^4x \, \partial_\mu \varphi \partial^\mu \varphi(x) = \int d^4x \, d^4y \; \varphi(x) \left[\partial^x_\mu \partial^\mu_y \delta^4(x-y)\right] \varphi(y)$$

$$Z_{free}(J) = \int \mathcal{D}\varphi \, \exp i\left[\int_x \int_y \varphi(x) \left[\frac{M_{xy}}{2}\right] \varphi(y) - \int_x J(x)\varphi(x)\right]$$

$$\sum \sum \quad \sum\sum \frac{\varphi M \varphi}{2} \qquad \sum J\varphi$$

$$= \exp \frac{i}{2} \int_x \int_y J(x) \left[M^{-1}_{xy}\right] J(y) \times \frac{(2\pi)^{\frac{\#\,sp\,pts}{2}}}{\mathrm{Det}^{1/2}\left[M_{xy}\right]}$$

$$= Z(0)$$

Wick's Theorem

$$\frac{1}{Z(0)} \frac{\partial^4 Z(J)}{\partial J(x_1) \partial J(x_2) \partial J(x_3) \partial J(x_4)} \bigg|_{J=0} = iM^{-1}(x_1 - x_2)\, iM^{-1}(x_3 - x_4) + iM^{-1}(x_1 - x_3)\, iM^{-1}(x_2 - x_4) + iM^{-1}(x_1 - x_4)\, iM^{-1}(x_2 - x_3)$$

M^{-1} is 2-pt function

2 pt func $\langle 0|T(\varphi(x)\,\varphi(y))|0\rangle$

$$= \frac{1}{Z(0)} \int \mathcal{D}\varphi \; \varphi(x)\varphi(y)\, e^{i(\cdots - J)}$$

$$= \frac{1}{Z(0)} \left(+i\frac{\partial}{\partial J(x)}\right)\left(+i\frac{\partial}{\partial J(y)}\right) Z(J) \Big|_{J=0} = M(x-y) = G(x-y)$$

$$\boxed{G(p) = \frac{i}{p^2 - m^2 + i\epsilon}}$$

Really ??

p-space $\mathcal{L} = \frac{1}{2}\varphi\,(p^2 - m^2)\,\varphi - J\varphi \;\longrightarrow\; J\,\frac{1}{p^2 - m^2}\,J$

$$M^{-1}_{ij}\, M_{jk} = \mathbb{1}_{ik} = \delta_{ik}$$

i, j discrete indices $\to$ Contin. Sp. time indices

$$M^{-1}(x-y) = \frac{1}{\partial_\mu \partial^\mu + m^2}$$

$\hbar = 1$

$$\int_y M^{-1}(x,y)\, M(y,z) = \delta^4(x-z)$$

$$(\partial_\mu \partial^\mu - m^2)^{-1} \delta^4() \quad (\partial_\mu \partial^\mu \delta^4(y-z))$$

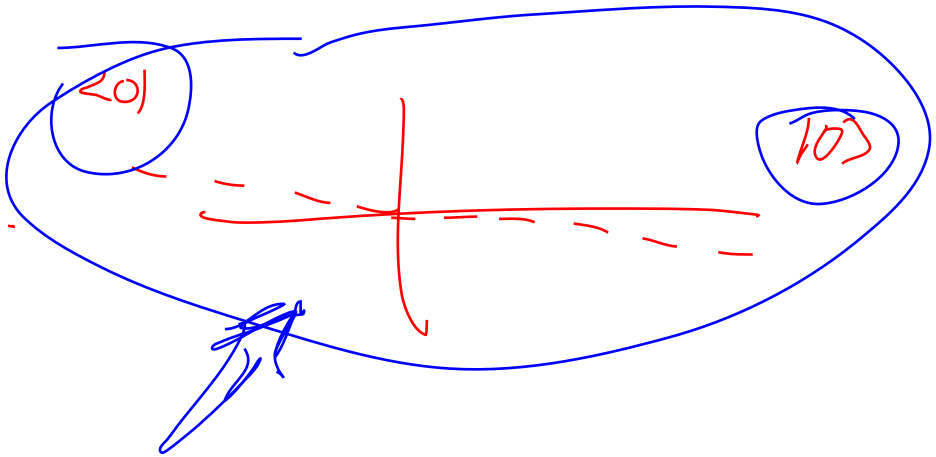
∂_z

$\partial_\mu \to -i p_\mu$ simpler

Let's transform this to p-space

But $\int dt \to \int dt(1-i\epsilon)$

And $\dfrac{d}{dt} \to \dfrac{d}{(1-i\epsilon)\,dt}$



$$G(p) = \int d^4x\, e^{ip\cdot x}\, G(x) \;\leftarrow\; \frac{\delta^2}{\delta J(x)\,\delta J(0)} \int \mathcal{D}\varphi\, e^{i\int d^3y\,dt\,[\;\;\;\varphi\;\;\;]}$$

$$(1-i\epsilon)\,\varphi(\nabla^2 - m^2)\varphi + \frac{1}{1-i\epsilon}\,\varphi(-\partial_t^2)\varphi - (1-i\epsilon)J\varphi$$

Can do $\int \mathcal{D}\varphi$ in p-space

$$\int d^3y\,dt\;\varphi(y,t)\,[\qquad]\,\varphi(y,t)$$

$$\int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot y}\,\tilde{\varphi}(q)$$

$$\nabla^2 \to -\vec{q}^{\,2}$$

$$\partial_t^2 \to -(q_0)^2$$

$$iS = i\int d^3x\,dt\; \varphi(x)\left[(1-i\epsilon)(\nabla^2 - m^2) - \frac{1}{1-i\epsilon}\partial_t^2\right]\varphi(x)$$

$$= \underline{\underline{i}}\int \frac{d^4q}{(2\pi)^4}\left[\frac{1}{2}(1-i\epsilon)\underbrace{(-m^2-\vec{q}^{\,2})}\tilde{\varphi}(q)\tilde{\varphi}(-q) + \frac{q_0^2}{1-i\epsilon}\frac{\tilde{\varphi}(q)\tilde{\varphi}(-q)}{2}\right]$$

$$\frac{i}{2}M(q)$$

$$\underline{\underline{M(q)}} = (1-i\epsilon)(-\vec{q}^{\,2}-m^2) + \frac{1}{1-i\epsilon}q_0^2$$

$$M^{-1} \simeq \frac{i}{q^2 - m^2 + i\epsilon}$$

$$\text{so } M^{-1}(q) = \frac{i}{\frac{1}{1-i\epsilon}q_0^2 - (1-i\epsilon)(\vec{q}^{\,2}+m^2)} = \frac{i}{q_0^2 - \vec{q}^{\,2} - m^2 + i\epsilon\underbrace{[q_0^2+\vec{q}^{\,2}+m^2]}_{\text{always }\geq 0}}$$

$$\frac{i\partial}{\partial J(x)}\frac{i\partial}{\partial J(y)} Z(J) = G(x-y)$$ — G_T or G_F correl of free thy.

$$\boxed{G(p) = \frac{i}{p^2 - m^2 + i\epsilon}}$$

But $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \boxed{- \frac{\lambda}{24}\varphi^4}$

interactions!!

Now what? Hope λ is small. Perturb.

$$\left[\int dx\, e^{-x^2/2}\, x^6\right] e^{Jx}\Big|_{J=0} = \left[\frac{\partial^6}{(\partial J)^6}\int dx\, e^{-x^2/2}\, e^{Jx}\right]_{J=0}$$

$$\frac{\partial}{\partial J}\int dx\, e^{-x^2/2}\, e^{Jx} \qquad \frac{\partial}{\partial J} e^{Jx} = x\, e^{Jx}$$

$$\left(\frac{\partial}{\partial J}\right)^6 e^{\frac{1}{2}J^2} = \text{pairs of } \frac{\partial}{\partial J}\text{'s} \qquad 6!! = 5\cdot 3\cdot 1 = 15$$

$$\int \prod_{i=1}^{N} dx_i\, e^{-\frac{1}{2}x_a M_{ab} x_b}\, x_{a_1}\cdots x_{a_n}\, e^{x_a J_a}\Big|_{J_a=0}$$

$$= \frac{\partial^n}{\partial J_{a_1}\cdots\partial J_{a_n}}\int \prod dx_i\, e^{-\frac{1}{2}xMx + xJ} = \frac{\partial^n}{\partial J_{a_1}\cdots\partial J_{a_n}}\, e^{\frac{1}{2}J M^{-1} J} = \sum_{\text{pairs}} M^{-1}_{a_{p_1}a_{q_1}}\, M^{-1}_{a_{p_2}a_{q_2}}\cdots$$

$$\int \mathcal{D}\varphi \quad \varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\exp\left[i\int -\varphi\,\partial_\mu\partial^\mu\varphi - m^2\varphi^2 + \varphi J\right]$$

$$\frac{-\partial^2 - m^2}{2}$$

inv. $\to G$

$$\underbrace{\frac{i\,\delta}{\delta J(x_1)}\cdots\frac{i\,\delta}{\delta J(x_n)}}$$

$$\left(\frac{i\,\delta}{\delta J(x_i)}\right)^n e^{-\frac{i}{2}\int_{xy} J(x)\,G(x-y)\,J(y)}$$

$J \to 0$

propagator

$$\mathcal{L} = \varphi[\ldots]\varphi$$

$$G(x-y) = [\ldots]^{-1}$$

$$= \sum_{\text{pairs}} G_{x_i x_j} G \cdots G_{xx}$$

Dirty Trick

$$Z(J) = \int \mathcal{D}\varphi \, \exp\left[i \int_x \underbrace{\varphi\left(-\frac{\partial_\mu \partial^\mu - m^2}{2}\right)\varphi - \varphi J}_{S_2} \underbrace{- \frac{\lambda}{24}\varphi^4}_{S_4} \right]$$

$$e^{A+B} = e^A e^B$$

$$e^{i \int S_{2\text{-fields}}} \; e^{-\frac{i\lambda}{24}\int d^4x\, \varphi^4(x)}$$

λ small: Taylor expand

$$\sum_{n=0}^{\infty} \left(\frac{-i\lambda}{24}\right)^n \frac{1}{n!} \left[\int d^4x\, \varphi^4(x)\right]^n$$

$$Z(J) = \sum_{n=0}^{\infty} \frac{\left(\frac{-i\lambda}{24}\right)^n}{n!} \int \mathcal{D}\varphi \, e^{i\varphi \frac{M}{2}\varphi - i\varphi J} \int_{x_1 \ldots x_n} \varphi^4(x_1) \cdots \varphi^4(x_n)$$

Just Do It

Now $\varphi(x) \to i\dfrac{\delta}{\delta J(x)}$

$4n$ deriv's

$$Z(J) = \sum_{n=0}^{\infty} \frac{\left(\frac{-i\lambda}{24}\right)^n}{n!} \int \mathcal{D}\varphi \int_{x_1 \cdots x_n} \underbrace{\left(\frac{i\delta}{\delta J(x_1)}\right)^4 \left(\frac{i\partial}{\partial J(x_2)}\right)^4 \cdots}\, e^{i\int \frac{\varphi(-\partial_\mu\partial^\mu - m^2)\varphi}{2} - \varphi J}$$

$$= \sum_n \frac{\left(\frac{-i\lambda}{24}\right)^n}{n!} \int_{x_1 \cdots x_n} \left(\frac{-i\partial}{\partial J(x_i)}\right)^{4n} \underbrace{\int \mathcal{D}\varphi\, e^{i\int\left[\frac{\varphi(-\partial_\mu\partial^\mu - m^2)\varphi}{2} - \varphi J\right]}}_{\to\, e^{+\frac{i}{2}\int_{xy} J(x) G(x-y) J(y)}}$$

$$\frac{\partial}{\partial J} = \text{combinations of } G\text{'s.}$$

$$\frac{\partial}{\partial J}\, Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\lambda}{24}\right)^n \underbrace{\int d^4x_1 \cdots d^4x_n \left(\frac{i^{4n}\,\partial^{4n}}{(\partial J(x_1))^4 \cdots (\partial J(x_n))^4}\right)}\, \underline{e^{\frac{i}{2}\int_{yz} J(y)\, G_F(y-z)\, J(z)}}$$

Do This Order-by Order in λ.

Let's DO Scattering $2\,\varphi,\ p_1 p_2 \to 2\varphi,\ k_1, k_2$

$$\sigma = \frac{1}{2p_1^0\, 2p_2^0\, |v_1 - v_2|} \int \frac{d^3k_1\, d^3k_2}{(2\pi)^6\, 2k_1^0\, 2k_2^0} (2\pi)^4 \delta^4(p_1+p_2-k_1-k_2)\, |\mathcal{M}_{pp\to kk}|^2$$

where

$$\int d^4x_1 \cdots d^4x_4\, e^{i(p_1\cdot x_1 + p_2\cdot x_2 - k_1\cdot x_3 - k_2\cdot x_4)} \langle 0|\, T(\varphi(x_1)\cdots\varphi(x_4))\,|0\rangle$$

$$= i\mathcal{M}(p_1 p_2, k_1 k_2)\,(2\pi)^4 \delta^4(p_1+p_2-k_1-k_2) \cdot \frac{i^4}{(p_1^2-m^2+i\epsilon)\cdots(k_2^2-m^2+i\epsilon)}$$

Lowest order : λ^0 : $\quad Z(J) = e^{-\frac{i}{2}\int_{x,y} J(x)\, G(x-y)\, J(y)}$

$$\langle 0|\, T\left(\varphi(x_1)\cdots\varphi(x_4)\right)|0\rangle = i^4\, \frac{\delta^4}{\delta J(x_1)\cdots J(x_4)}$$

$\delta^4(\cancel{p_1+p_2})$

$(2\pi)^4\delta^4(p_1-k_1)\,G(p_1)$

$(2\pi)^4\delta^4(p_2-k_2)\,G(p_2)$

$k_1=p_1 \qquad k_2=p_1$

$$= i^6\Big(G(x_1-x_2)G(x_3-x_4) + G(x_1-x_3)G(x_2-x_4) + G(x_1-x_4)G(x_2-x_3) \Big)$$

$$\int d^4x_1\, d^4x_3\, e^{i(p_1\cdot x_1 - k_1\cdot x_3)}\, G(x_1-x_3) = \int d^4r\, d^4c\, e^{ic\cdot(p_1-k_1)}\, e^{ir\cdot\left(\frac{p_1+k_1}{2}\right)}\, G(r)$$

$x_1-x_3=r \qquad d^4x_1\,d^4x_3 = d^4r\,d^4c$

$\frac{x_1+x_3}{2} = c$

$(2\pi)^4\delta^4(p_1-k_1) \qquad p_1=k_1 \qquad G(p_1)$

$$= (2\pi)^4\delta^4(p_1-k_1)\,G(p_1)$$

p_1 $\longrightarrow$ p_2 $\longleftarrow$

Huge chance $\longleftarrow$ $\longrightarrow$ $k_1 = p_1$

$k_2 = p_2$

Not Scattering

Important but Boring

Order λ^1

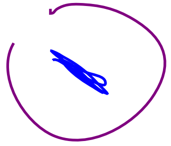
$$\int d^4x_1 \cdots d^4x_4 \, e^{i(p_1\cdot x_1 + p_2\cdot x_2 - k_1\cdot x_3 - k_2\cdot x_4)} \frac{i^4 \delta^4}{\delta J(x_1)\cdots\delta J(x_4)} \left(-\frac{i\lambda}{24}\int d^4y \left(\frac{i\delta}{\delta J(y)}\right)^4\right) e^{-\frac{i}{2}\int_{z_1 z_2} J(z_1)G(z_1-z_2)J(z_2)}$$

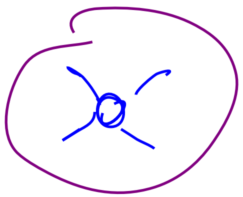
$8\,\frac{\partial}{\partial J}$'s. $7\cdot5\cdot3\cdot1$ pairings

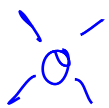
105 terms.

105 ways to pair off $(x_1\ x_2\ x_3\ x_4\ y\ y\ y\ y)$

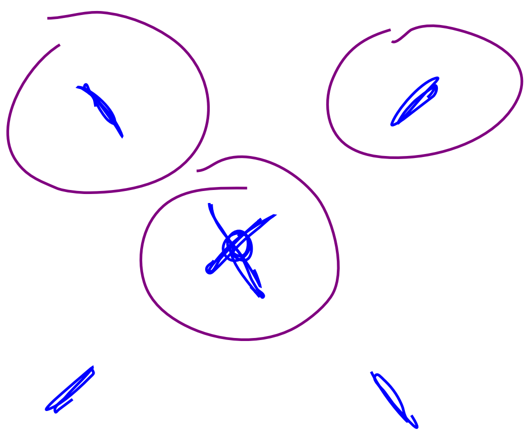
Graphical approach

$\frac{i\delta}{\delta J(x)}$ write it as a  begining/end of line

$\int d^4y \left(\frac{i\partial}{\partial J(y)}\right)^4$  $\circ \int d^4y$

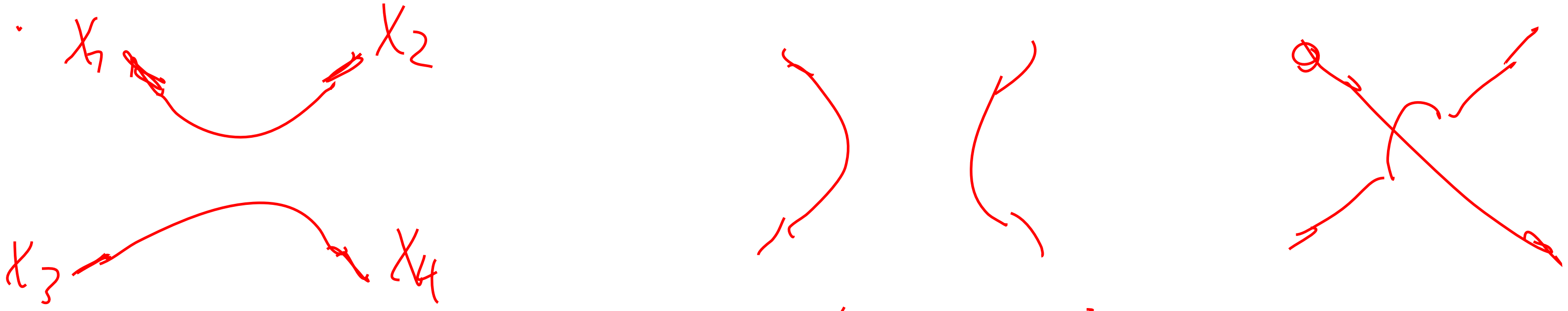
 $\frac{\partial}{\partial J}$

$\underbrace{\underbrace{\partial\partial\partial\partial} \int_y \partial\partial\partial\partial\, e^{\frac{1}{2}JGJ}}$

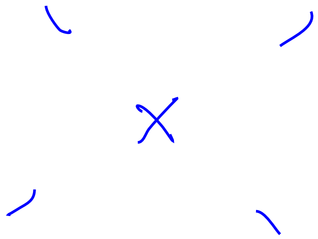


Count ways to pair

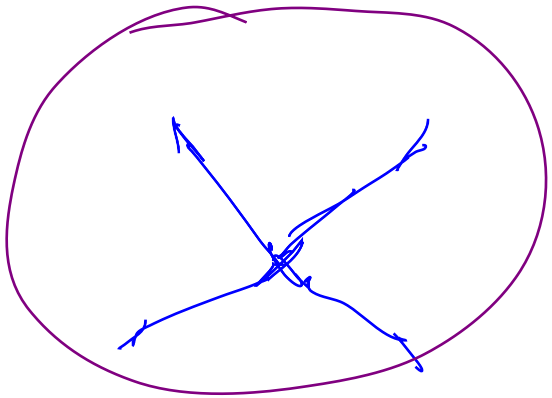
λ^0:



$G(x_1-x_2)G(x_3-x_4)$ $G(x_1-x_3)G(x_2-x_4)$ . . . .



7 distinct classes

$\rightarrow$ 

$$(i)^{\#} \int d^4y \; G(x_1-y)G(x_2-y)G(x_3-y)G(x_4-y)$$

$$\times \frac{(-i\lambda)}{24} \cancel{24} \rightarrow -i\lambda$$

$$\dashrightarrow \mu = \lambda$$

$4 \cdot 3 \cdot 2 \cdot 1 = 24$

Fourier transform it!

$\delta^4(p_1-q)$

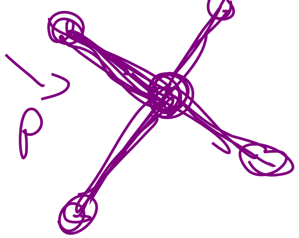
$$-i\lambda \int d^4x_1 \cdots d^4x_4 \int d^4y \; e^{i(p_1 x_1 \cdots - p_2 x_4)}$$

$G(p_1)G(p_2)G(-k_1)G(-k_2)$

$G(x_1-y) \cdots G(x_4-y)$

$(2\pi)^4 \delta^4(p_1+p_2-k_1-k_2)$

$$G(x_1-y) = \int \frac{d^4q_1}{(2\pi)^4} e^{-iq\cdot(x_1-y)} G(q)$$

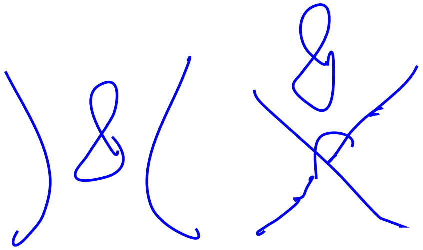


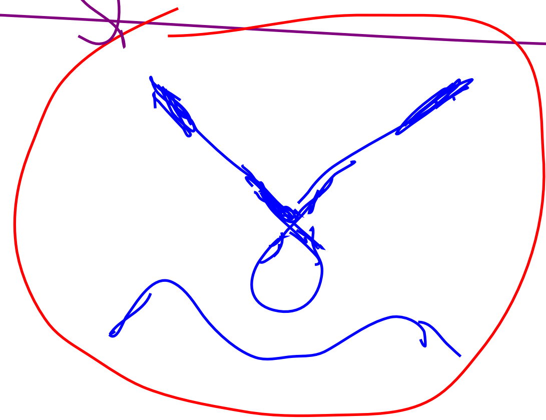
$(-i\lambda)(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\,G(p_1)G(p_2)G(-k_1)G(-k_2)$

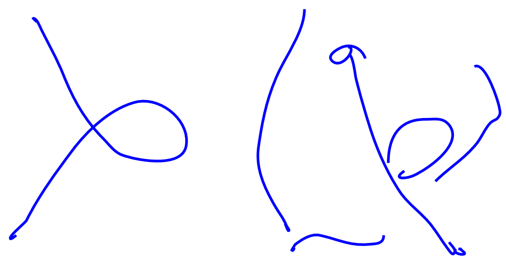
$-iM$

$\boxed{M=\lambda}$

$\frac{i}{p_1^2-m^2+i\epsilon}\quad\frac{i}{p_2^2-m^2+i\epsilon}\quad\frac{i}{k_1^2-m^2+i\epsilon}\quad\frac{i}{k_2^2-m^2+i\epsilon}$







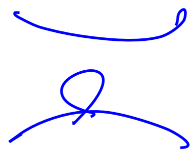
$3\int d^4y\,G^2(y-y)\,G(x_1-x_2)G(x_3-x_4)$

$+3$

$+3$ / 9

free prop.

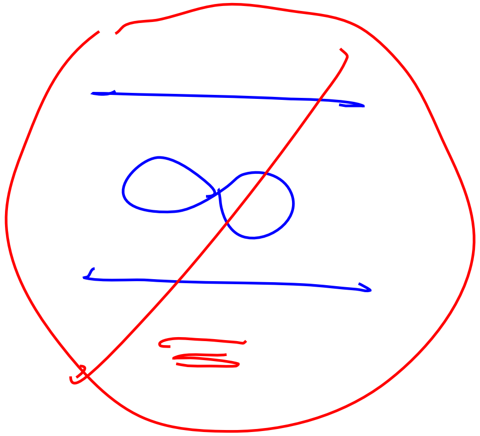
$4\cdot 3\cdot$



6 ways

72 terms

$9+72+24=105$

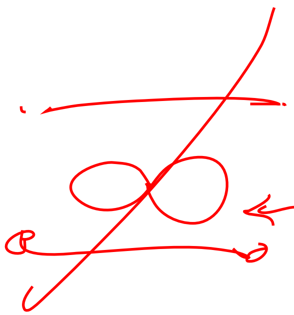


$\int d^4y\, G(y-y)\,G(y-y)$ huh?

$$\langle 0|\, T(\varphi\varphi\varphi\varphi)\,|0\rangle = \frac{1}{Z(0)}\,\frac{\delta^4 Z(J)}{\delta J \dots}\Big|_{J=0}$$

λ^0 λ^1

λ λ^0



$$-\frac{i\lambda}{24}\int d^4y \left(\frac{\partial}{\partial J(y)}\right)^4 e^{\frac{i}{2} J\Delta J}\Big|_{J=0}$$

$$\frac{1}{1+\lambda} = 1 - \lambda + \dots$$

Disconnected Bubble

Bubble from $\frac{1}{Z(0)}$ cancel bubbles in expanding $Z(J)$