Lorentz & Translation Symmetries in QFT

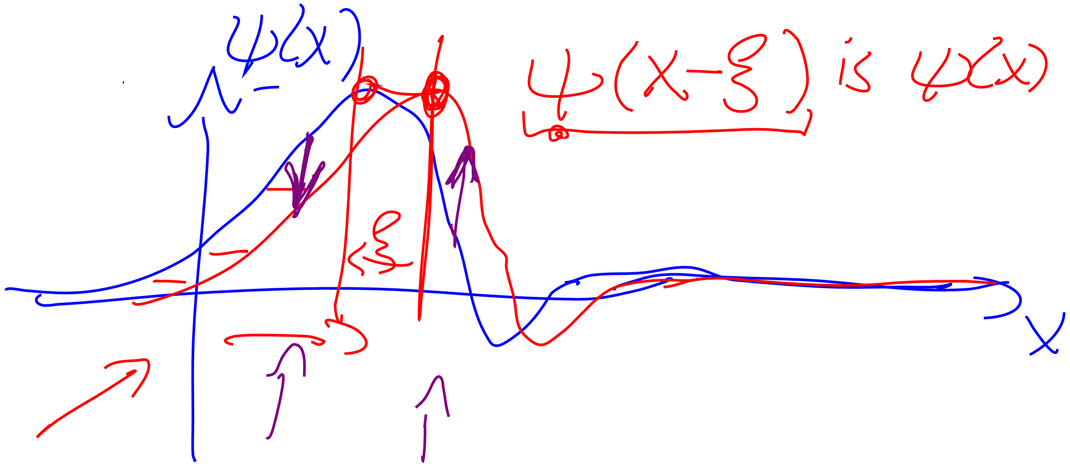
1) Reminder : symmetry transforms
   - generators — QM version
   - representations

2) The story for rotations, to warm you up

3) Lorentz — larger version

How translations work

Translate distance ξ



$\underbrace{\psi(x-\xi)}$ is $\psi(x)$ shifted by ξ

$$\underbrace{\tilde{\psi}}_{\text{new func}}(x) = \psi(x-\xi)$$

Func goes down where $\dfrac{d\psi}{dx} > 0$

up where $\dfrac{d\psi}{dx} < 0$

$$\psi(x) = \langle x|\psi\rangle$$

$$\langle x| \hat{T}(\xi) \underbrace{|\psi\rangle}_{\text{shifted state}}$$

$\boxed{T(\xi)}$ translation op.

For small ξ,

$$\underline{T(\xi)\psi(x)} = \psi(x-\xi) = \psi(x) - \xi\frac{d}{dx}\psi(x) + O(\xi^2)$$

$\hat{T}(\xi)$ operator. $\xi = 0$: $T(\xi) = \mathbb{1}$

ξ tiny $\quad \hat{T}(\xi) = \mathbb{1} + \xi \left(\frac{-i}{\hbar}\right)\hat{p} + O(\xi^2)$ $\quad \frac{-i}{\hbar}$ is so $\hat{p}$ is standard p-op.

$-i$?? $\hat{p}$ should be Hermitian

$\hat{T}$ should be unitary

$N\xi \sim 1$

$\rightarrow N\xi^2$ is tiny

$$\hat{T}(2\xi) = T(\xi)T(\xi)$$

$$T(N\xi) = (T(\xi))^N = \left(1 - \frac{i}{\hbar}\xi p + O(\xi^2)\right)^N \xrightarrow{N} \exp\left[\frac{-iN\xi}{\hbar} p\right]$$

$$T^\dagger(\xi) = 1^\dagger - \left(\frac{i}{\hbar}\right)^\dagger \xi \hat{p}^\dagger = 1 + \frac{i\xi}{\hbar}\hat{p} = T(-\xi) \qquad \underbrace{T^\dagger = T^{-1}}_{\text{unitary}}$$

$(i)^\dagger = -i \qquad p^\dagger = p$

$$\langle \psi_1 | \quad \mathcal{O} \quad | \psi_2 \rangle$$

$\hat{X}$ for instance

shift states forward.

$$|\psi_2\rangle \to T(\xi)|\psi_2\rangle$$

$$\langle \psi_1 | \to \langle \psi_1 | T^\dagger(\xi)$$

$$= \langle \psi_1 | T(-\xi)$$

$$\underbrace{\langle \psi_1 |}\ \underbrace{\overbrace{T(-\xi)\,\mathcal{O}}\ \overbrace{T(\xi)\,|\psi_2\rangle}}\ \underbrace{|\psi_2\rangle}$$

$$\tilde{\mathcal{O}}(x) = \mathcal{O}(x-\xi)$$

$$\langle x | \hat{T}^\dagger \hat{X} \hat{T} | \psi \rangle = x\,\psi(x)$$

$$= (x-\xi)\,\psi(x-\xi)$$

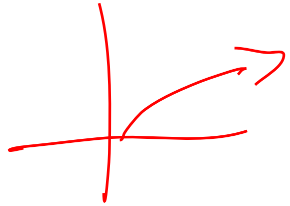
Similarly

$$x_i \to R_{ij} x_j \quad \text{rotated point}$$

$$\varphi(x) \to \tilde{\varphi}(x) = \varphi(R^{-1}_{ij} x_j)$$

4-Translation $T(\xi) = 1 - \frac{i}{\hbar}\hat{P}^{\mu}\xi^{\nu}g_{\mu\nu}$

$\hat{P}_i$ vector of transl. gen. op.'s



$P^0 = H$ time-trans.

$\vec{P} = \vec{P}$ sp. trans. op.

Rotations $U(\Theta_i) = 1 - \frac{i\Theta_i}{\hbar}\hat{J}_i$

infinitesimal θ

Θ_i : extents of rot.

$\hat{J}_i$: op's carry out rot's.

$\hat{\Theta}$ axis you rotate about

$|\Theta|$ angle

Finite Θ: $U(\Theta_i) = \exp\left[\frac{-i\Theta_i\hat{J}_i}{\hbar}\right]$

Dim'less

I expect $\varphi(x) \longrightarrow T^{-1}(\xi)\,\varphi(x)\,T(\xi) = \varphi(x-\xi)$

$\longrightarrow U(\theta)\,\varphi(x)\,U(\theta) = \varphi(R^{-1}_{ij}x_j)$

If φ is scalar

vector? $A_i \longrightarrow R_{ij}A_j$ not A_i

$$U(-\theta)A_i U(\theta) = R_{ij}A_j(R^{-1}x)$$

$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$ 3 op's. Rot's, these mix

In general, for fields φ_a $a = 1, \dots,$

$$U(-\theta_i)\,\varphi_a\,U(\theta_i) = M_{ab}(\theta)\,\varphi_b(R^{-1}x)$$

Properties of M's: They Represent rotations!

2 Rot matrices $\underbrace{R_1 \quad R_2}$ eg $R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$ θ_1 rot about x ax.

$X \to R_1 X \to R_2(R_1 X) = R_2 R_1 X$ $\quad R_2 = \begin{pmatrix} \cos\theta_2 & 0 & +\sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$ θ_2 rot about y ax.

$|\psi\rangle \to \mathcal{U}(R_1)|\psi\rangle \to \mathcal{U}(R_2)\,\mathcal{U}(R_1)|\psi\rangle$

$$\varphi_a \xrightarrow{R_1} \mathcal{U}^{-1}(R_1)\,\varphi_a\,\mathcal{U}(R_1) = M_{ab}(R_1)\,\varphi_b(R_1^{-1}X) = \tilde{\varphi}_a$$

$$\tilde{\varphi}_a \xrightarrow{R_2} \mathcal{U}^{-1}(R_2)\,\tilde{\varphi}_a\,\mathcal{U}(R_2) = M_{ab}(R_2)\,M_{bc}(R_1)\,\varphi_c(R_1^{-1}R_2^{-1}X)$$

But I could also just apply $R_2 R_1$

$\mathcal{U}(R_2)\,\mathcal{U}(R_1) = \mathcal{U}(R_2 R_1)$. $\quad \mathcal{U}^{-1}(R_2 R_1) \overset{(\mathcal{U}^{-1})}{=} \mathcal{U}^{\dagger}(R_1)\,\mathcal{U}^{\dagger}(R_2)$

$$\varphi_a \to \mathcal{U}^{-1}(R_2 R_1)\,\varphi_a\,\mathcal{U}(R_2 R_1) = M_{ac}(R_2 R_1)\,\varphi_c(R_1^{-1}R_2^{-1}X)$$

$$M_{ac}(R_2 R_1) = M_{ab}(R_2)\,M_{bc}(R_1)$$

Matrices descr. how op-components change under Rot (or other trans):

$$M_{ab}(R_1)\, M_{bc}(R_2) = M_{ac}(R_1 R_2)$$

spin 0 1×1

spin 1/2 2×2

spin 1 3×3

$\vdots$

M's must represent the rotations R.

What M's do that?

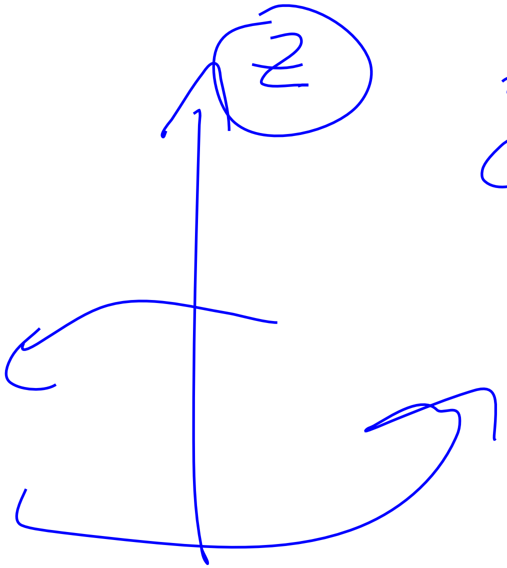
Most general $M_{ab} = \bigoplus$ irreducible M's

$$M_{ab} = \begin{pmatrix} \boxed{\phi} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{\begin{matrix} & \\ & \end{matrix}} & & 0 & 0 \\ 0 & 0 & \boxed{\begin{matrix} & & \\ & & \\ & & \end{matrix}} & & 0 \\ 0 & 0 & 0 & & \boxed{\phi} \end{pmatrix} \begin{pmatrix} \\ \\ \varphi_a \\ \\ \\ \end{pmatrix}$$

$\varphi_a \to U_{ab}\varphi_b$ unitary basis chg.

$M \to U^\dagger M U$

How did it work for Rot's?



z stays same

(x, y) mix

~~z-axis~~ rot

xy plane rotation

x, y mix

z, w stay same, $x\,y$ mix

z, w, u " " x, y mix

$R_i,\ J_i,\ \Theta_i \longrightarrow$

$R_{ij}\quad J_{ij}\quad \Theta_{ij}$ plane which is changing

$$\begin{bmatrix} 1 & -\omega \\ \omega & 1 \end{bmatrix} = \delta_{ij} + \omega_{ij}$$

infinitesimal

$$\omega_{ij} = -\omega_{ji}$$

Usually define $\Theta_i = \frac{1}{2}\epsilon_{ijk}\,\omega_{jk}$

$$\Theta_z = \frac{1}{2}\left(\epsilon_{zxy}\,\omega_{xy} + \epsilon_{zyx}\,\omega_{yx}\right)$$

$$J_i = \frac{1}{2}\epsilon_{ijk}\,J_{jk}$$

Similarly

$$U(\theta) = 1 - \frac{i\theta_i J_i}{\hbar}$$

$$\rightarrow \qquad = 1 - \frac{i}{2\hbar}\omega_{ij} J_{ij}$$

$$[J_i, J_j] = i\hbar\,\epsilon_{ijk} J_k$$

looks like what?

$$[J_{ij}, J_{lm}] = i\hbar\left(\delta_{im} J_{lj} + \delta_{jm} J_{il} - \delta_{il} J_{mj} - \delta_{jl} J_{im}\right)$$

yeesh!

why?

Consider 2 finite trans

$$\exp\left[-i \frac{J_{ij}\,\omega^1_{ij}}{2\hbar}\right] \exp\left[-i \frac{J_{ij}\,\omega^2_{ij}}{2\hbar}\right]$$

ω^1, ω^2 descr. 2 rot's, eg, ($\omega^1_{xy} \neq 0$, $\omega^2_{xz} \neq 0$)

$$\lim_{N \to \infty} \left(1 - \frac{iJ_{ij}\omega^1_{ij}}{2\hbar N}\right)^N \left(1 - \frac{iJ_{ij}\,\omega^2_{ij}}{2\hbar N}\right)^N$$

Is This $(\quad \omega^2)(\quad \omega^1)$?? no. why not?

$$(\quad)(\quad)(\quad)(\omega^1)(\quad)(\quad)(\quad)(\quad)(\omega^1)(\omega^2)_N$$

N $\qquad$ N

N^2 order changes.

N obj. each passing N obj.

$$\left(1 - \frac{iJ_{ij}\omega^1_{ij}}{2\hbar N}\right)\left(1 - \frac{iJ_{ij}\omega^2_{ij}}{2\hbar N}\right)$$

$$= (\quad)(\quad) + \frac{1}{N^2}$$

Need to understand

$$\left(1-\frac{i}{2\hbar N}J_{ij}\,\omega^1_{ij}\right)\left(1-\frac{i}{2\hbar N}J_{lm}\,\omega^2_{lm}\right) \text{ to } \mathcal{O}(1/N^2)$$

Applies R_1 Applies R_2 $\to R_1R_2$

opposite order $\to R_2R_1$

$$(R_1)_{ij} = \delta_{ij} + \omega^1_{ij} + \frac{1}{2}\omega^1_{ik}\omega^1_{kj} + \cancel{\ldots}$$

$$(R_2)_{ij} = \delta_{ij} + \omega^2_{ij} + \frac{1}{2}\omega^2_{ik}\omega^2_{kj} + \cancel{\ldots}$$

$$(R_1R_2)_{ij} = \left(\delta_{ik}+\omega^1_{ik} + \omega\omega\right)\left(\delta_{kj}+\omega^2_{kj}+\omega\omega\right) = \delta_{ij}+\omega^1_{ij}+\omega^2_{ij} + \omega^1_{ik}\omega^2_{kj} + \omega^1\omega^1 + \omega^2\omega^2$$

$$(R_2R_1)_{ij} = (\omega^2\to\omega^1 \quad \omega^1\to\omega^2) = \delta_{ij}+\omega^2_{ij}+\omega^1_{ij}+\omega^2_{ik}\omega^1_{kj} + \omega^2\omega^2 + \omega^1\omega^1$$

$$\boxed{(R_2R_1)_{ij}} = \boxed{(R_1R_2)_{ij}} + (\omega^2_{ik}\omega^1_{kj} - \omega^1_{ik}\omega^2_{kj})$$

Antisymm in $i \leftrightarrow j$

$$U_2U_1 = U_1U_2 + \text{extra rotation}$$

Extra rot. by "angle" $(\omega^2\omega^1 - \omega^1\omega^2)$

$$U_2U_1 = U_1U_2 + \frac{-i\hat{J}_{ij}}{2\hbar}(\omega^2_{ik}\omega^1_{kj} - \omega^1_{ik}\omega^2_{kj})$$

$$\left(1 - \frac{i\hbar}{2}J_{ij}\omega^2_{ij}\right)\left(1 - \frac{i\hbar}{2}J_{lm}\omega^1_{lm}\right) = (\omega^2 \leftrightarrow \omega^1) + \ldots$$

$$1 - \frac{i\hbar}{2}J(\omega^2+\omega^1) + \left(\frac{-i\hbar}{2}\right)^2 J_{ij}J_{lm}\,\omega^2_{ij}\omega^1_{lm} = (\text{same but } J_{lm}J_{ij}$$

$$U_2U_1 - U_1U_2 \quad \left(\frac{-i}{2\hbar}\right)^2 (J_{ij}J_{lm} - J_{lm}J_{ij})\,\omega^2_{ij}\omega^1_{lm} = \frac{-i}{2\hbar}J_{ij}(\omega^2_{ik}\omega^1_{kj} - \omega^1_{ik}\omega^2_{kj})$$

$$\left(\cancel{\frac{-i}{2\hbar}}\right)^2 [\overset{1}{J}_{ij}, \overset{1}{J}_{lm}] \underbrace{\overset{2}{w}_{ij} \overset{1}{w}_{lm}} = \cancel{\frac{-1}{2\hbar}} J_{rs} \left(\underbrace{\overset{2}{w}_{rt} \overset{1}{w}_{ts}} - \underbrace{\cancel{\overset{1}{w}_{rt} \overset{2}{w}_{ts}}}_{-\overset{2}{w}_{ts} \overset{1}{w}_{rt}}\right)$$

$2\hbar i$

$$[J_{ij}, J_{lm}] \, \overset{2}{w}_{ij} \overset{1}{w}_{lm} = \left(J_{rs} \overset{2}{w}_{rt} \overset{1}{w}_{ts} - J_{rs} \overset{2}{w}_{ts} \overset{1}{w}_{rt} \right) 2i\hbar$$

$$J_{rs} \delta_{tw} \overset{2}{w}_{rt} \overset{1}{w}_{ws} - J_{rs} \delta_{tw} \overset{2}{w}_{ts} \overset{1}{w}_{rw}$$

$$[J_{ij}, J_{lm}] \overset{2}{w}_{ij} \overset{1}{w}_{lm} = 2i\hbar \left(J_{im} \delta_{jl} \overset{2}{w}_{ij} \overset{1}{w}_{lm} - J_{lj} \delta_{im} \overset{2}{w}_{ij} \overset{1}{w}_{lm} \right)$$

$$\frac{J_{im} - J_{mi}}{2} \qquad \frac{J_{lj} - J_{jl}}{2}$$

$ij \to -ji$

$lm \to -ml$

Lorentz Transforms?

$\delta_{ij} \to g_{\mu\nu}$

$\exp -i \frac{J_{ij}}{2\hbar} \omega_{ij} \to \exp -i \frac{J^{\mu\nu}}{2\hbar} \omega_{\mu\nu}$ $\qquad i \to \mu$

$\delta \to g$

$$[J^{\mu\nu}, J^{\alpha\beta}] = i\hbar \left(g^{\nu\alpha} J^{\mu\beta} + g^{\mu\alpha} J^{\beta\nu} - g^{\mu\beta} J^{\alpha\nu} - g^{\nu\beta} J^{\mu\alpha} \right)$$

sign opposite $\quad g_{ij} = -\delta_{ij}$

μ, ν space: $J_{ij} \to J_i = \frac{1}{2} \epsilon_{ijk} J_k$ as before.

J^{01}: x-axis L.T. accel in i-direction $x_i \leftrightarrow t$

work $\to$ $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$

$\mu, \nu \to i, 0$?? $\quad K^i \equiv J^{0i}$ generator of boosts

$$\exp \frac{-i}{2\hbar}\omega_{\mu\nu}J^{\mu\nu} = \exp\left(-i\theta_i J_i - i b_i K_i\right)$$

J_i, K_i

θ_i: rot angles about x y z axes

b_i: γv's along x y z axes

b_i for small vel. is $\frac{v_i}{c}$

$$\boxed{[J_i, J_j] = i\hbar\epsilon_{ijk}J_k}$$

$$[K_i, J_j] = i\hbar\epsilon_{ijk}K_k$$

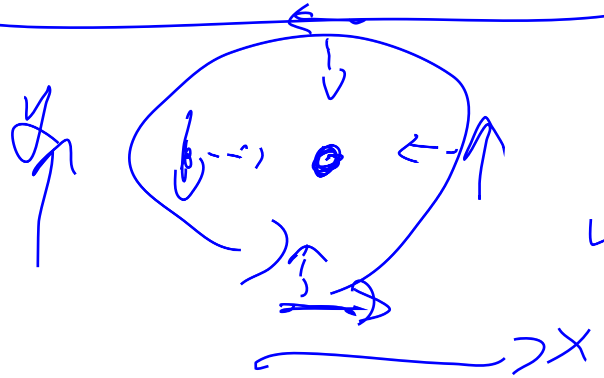
$$\boxed{[K_i, K_j] = -i\hbar\epsilon_{ijk}J_k}$$

J is a vector

K is a vector $\quad [A_i, J_j] = i\hbar\epsilon_{ijk}A_k$

commute 2 boosts, get a rotation!!

Thomas Precession



$\underbrace{y, -x}, -y, +x \rightarrow z$-rotation