Lecture 15: Lorentz Group & Representations

Last time: $\underline{x}^\mu \xrightarrow{\text{trans}} x^\mu + \xi^\mu$

$\underline{x}^\mu \xrightarrow{\text{Lorentz}} \Lambda^\mu{}_\nu x^\nu \underset{\text{Infinitesimal}}{\simeq} x^\mu + \omega^\mu{}_\nu x^\nu$

Infinitesimal: $(\omega_{\mu\nu} = -\omega_{\nu\mu})$

States transform

$|\psi\rangle \longrightarrow T(\xi)|\psi\rangle$ — unitary op's

apply $x \to x+\xi$, Λx

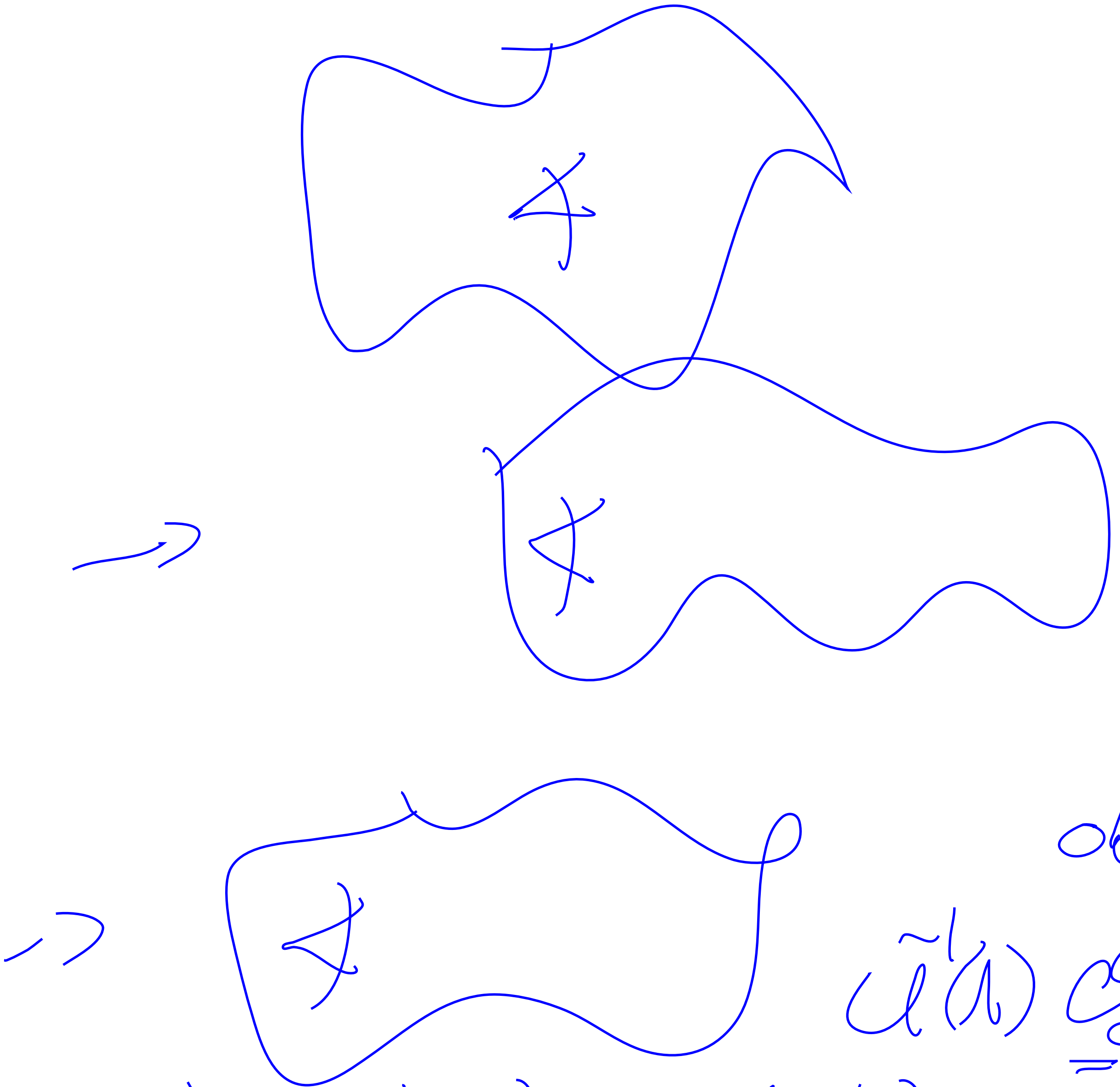
or $\mathcal{U}(\Lambda)|\psi\rangle$

on states

$$\langle\psi|\mathcal{O}|\psi\rangle \longrightarrow \langle\psi|\underbrace{T^{-1}\mathcal{O}\,T}_{\text{Transformed op.}}|\psi\rangle$$

$\psi(x) \to \psi(x+\xi)$

or $\mathcal{O}(x) \to \mathcal{O}(x-\xi)$



state forward

$\parallel$

observer back

$U^{-1}(\Lambda)\,\phi_a(x)\,U(\Lambda)$

$M_{ab}(\Lambda)\,\phi_b(\Lambda^{-1}x)$

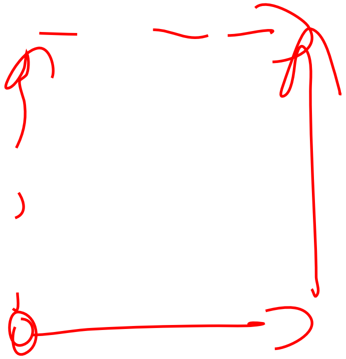
$=$

$M(\Lambda_1)\,M(\Lambda_2) = M(\Lambda_1\Lambda_2)$

M's represent Λ's.

$$\mathcal{U}(\Lambda) = \mathcal{U}(\omega_{\mu\nu}) = 1 - \frac{i}{2\hbar}\omega_{\mu\nu}\boxed{\hat{J}^{\mu\nu}} \to \text{Ang. mom gen. of Lorentz}$$

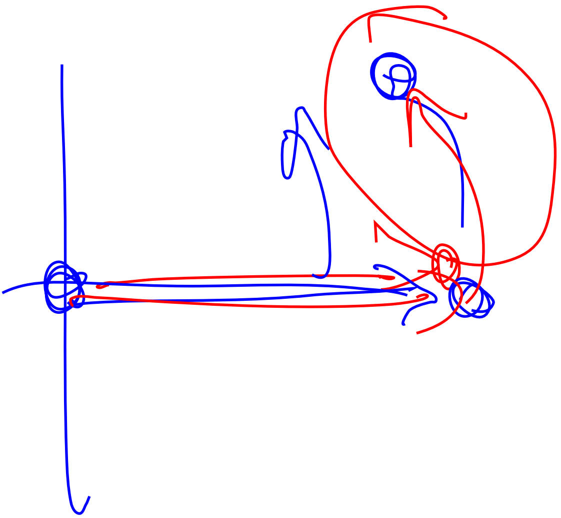
$$T(\xi) = 1 - \frac{i}{\hbar}\xi_\mu \boxed{\hat{p}^\mu} \to \text{momenta gen. of transl.}$$



$$T(\xi_1)T(\xi_2) = T(\xi_2)T(\xi_1)$$

$$[p^\mu, p^\nu] = 0$$

Translate then rotate $\neq$ rotate, then translate



$$x^\mu \xrightarrow[T(\xi)]{\xi} (x^\mu + \xi^\mu) \xrightarrow[\mathcal{U}(\omega)]{\omega} (\delta^\mu_\nu + \omega^\mu{}_\nu)(x^\nu + \xi^\nu)$$

$$= x^\mu + \xi^\mu + \omega^\mu{}_\nu x^\nu + \omega^\mu{}_\nu \xi^\nu$$

$$x^\mu \xrightarrow{\omega} (\delta^\mu_\nu + \omega^\mu{}_\nu)x^\nu \xrightarrow{\xi} x^\mu + \xi^\mu + \omega^\mu{}_\nu x^\nu + 0$$

$$T(\xi)\mathcal{U}(\omega) \neq \mathcal{U}(\omega)T(\xi) \quad \text{Differ by transl.}$$

$$[p^{\mu}, J^{\alpha\beta}] = -\hbar i\,(g^{\mu\alpha} p^{\beta} - g^{\mu\beta} p^{\alpha})$$

p^x $\quad J^{xy}$ $\quad J^{1} \quad p^y$

$p^y \to -p^x$

$$[J^{\mu\nu}, J^{\alpha\beta}] = i\hbar\,(g^{\mu\alpha} J^{\beta\nu} - g^{\mu\beta} J^{\alpha\nu} + g^{\nu\alpha} J^{\mu\beta} - g^{\nu\beta} J^{\mu\alpha})$$

$J_i = \epsilon_{ijk} J^{jk}$

3-index $\quad$ 3D tensor 4-index. $\quad$ Rot.

$K_i = J^{0i}$ $\to$ sp. time mixing "boost" trans.

$$[J_i, J_j] = i\hbar\,\epsilon_{ijk} J_k$$

$$[K_i, J_j] = i\hbar\,\epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\hbar\,\epsilon_{ijk} J_k$$

$$[P_i, J_j] = i\hbar\,\epsilon_{ijk} P_k$$

$$[P_i, K_j] = \pm i\hbar\,\delta_{ij}\, P^0 \quad (?)$$

$$[P^0, J_i] = 0$$

$$[P^0, K_i] = \pm i\hbar\, P_i$$

Transform laws for gen of $\boxed{SO(3,1)}$ 3-sp. 1 time $\quad g_{\mu\nu} = \begin{bmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$

$SO(N,M)$

$$\boxed{M_i = \frac{J_i + iK_i}{2}} \qquad \boxed{N_i = \frac{J_i - iK_i}{2}}$$

(L_i in some people's notation) $\quad$ (R_i in some notations.)

$$[M_i, M_j] = i\epsilon_{ijk} M_k$$

$$[N_i, N_j] = i\epsilon_{ijk} N_k$$

$$[M_i, N_j] = 0$$

$$J_i = M_i + N_i$$

$$K_i = \boxed{i}(N_i - M_i)$$

J, K split into 2 subsets, mutually commute!

$$SO(4) = SO(3,1) = SU(2) \otimes SU(2)$$

So what about fields?

$$\{\varphi_a\} \qquad \bar{U}(\omega)\,\varphi_a\,U(\omega) = M_{ab}(\omega)\,\varphi_b$$

$$\left.\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}\right\}$$

$$U \simeq 1 - \frac{i\,\omega_{\mu\nu}}{2\hbar} J^{\mu\nu}$$

$$M_{ab} \simeq \delta_{ab} - i\,\frac{\omega_{\mu\nu}}{2}\,\mathcal{J}^{\mu\nu}_{ab}$$

Matrix on set of fields

Matrix for each $(\mu\nu)$ pair

M's represent group:

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}] = i g^{\mu\alpha}\mathcal{J}^{\beta\nu} - i g^{\mu\beta}\mathcal{J}^{\alpha\nu} + i g^{\nu\alpha}\mathcal{J}^{\mu\beta} - i g^{\nu\beta}\mathcal{J}^{\mu\alpha}$$

$$\mathcal{J}^{\mu\nu}_{ab} = g^{\mu}_{a}\, g^{\nu}_{b} \;:\; M_{\mu\nu} = g_{\mu\nu} - i\,\omega_{\mu\nu}$$

N What are most general $J^{\mu\nu}_{ab}$? J^i_{ab} K^i_{ab}

$$M^i_{ab} = \frac{J^i + iK^i}{2} \qquad N^i_{ab} = \frac{J^i - iK^i}{2}$$

$$[M_i, N_j] = i\epsilon_{ijk}\, 0$$

$$\Rightarrow [M_i, M_j] = i\epsilon_{ijk} M_k$$

$$\Rightarrow [N_i, N_j] = i\epsilon_{ijk} N_k$$

M, N obey comm rel.

Just like Ang. Mom. Representations.

M_i: a rep of $SU(2)$

N_i: another rept of $SU(2)$

$\hat{J}_i$ not gen. $\mathcal{J}^i_{ab}\,\phi_b$

Scalar: $\phi_a \to \phi_a = (\delta_{ab} + \theta)\phi_b$

$\mathcal{J}^i_{ab} = [0]$ scalar, trivial rep.

$\mathcal{J}^i_{ab} = \frac{1}{2}\sigma^i$

$$\boxed{\left[\tfrac{1}{2}\sigma^i, \tfrac{1}{2}\sigma^j\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}}$$

Possible transform rule

$$J^x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & \\ 1 & 0 & 1 \\ & 1 & 0 \end{bmatrix} \quad J^y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad J^z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

spin-1 Rep.

$M_i = [0]$ $\qquad$ fine choice $\qquad$ $J_i = M + N$

$N_i = \frac{\sigma^i}{2}$ $\qquad\qquad$ $K_i = i(N - M)$

$$\boxed{J_i = \frac{\sigma_i}{2}} \qquad + \qquad \boxed{K_i = i\frac{\sigma_i}{2}}$$

2-comp. set of states/op's

$\psi^a \quad a = 1,2$

Rot. matrices

$[J_i, J_j] = i\epsilon\, J_k$,

$[J, K] = i\, K$

$[K, K] = -i\, J$

creates spin-1/2 particles

Right-Weyl Rep.

$M_i = \sigma_i/2$ $\qquad \longrightarrow \qquad$ $J_i = \frac{\sigma_i}{2}$ $\longleftarrow$ spin-1/2 part's

$N_i = 0$ $\qquad\qquad\qquad$ $K_i = -i\sigma_i/2$

Left-Weyl Rep.

M_i : spin $-\frac{m}{2}$ rep. of rot.

N_i : spin $-\frac{n}{2}$ rep of rot

$\left(\frac{m}{2}, \frac{n}{2}\right)$ rep.

$(0,0)$: $M_i = [0]$ $N_i = [0]$ $M_{ab} = [1]$ scalars

$\left(\frac{1}{2}, 0\right)$: $M_i = \frac{\sigma_i}{2}$ $N_i = 0$ $J_i = \frac{\sigma_i}{2}$ $\begin{bmatrix} \sigma/2 & 0 \\ 0 & \sigma/2 \end{bmatrix}$

$K_i = -i\sigma_i/2$

$\boxed{\psi_\alpha} \xrightarrow{\; u^{-1}\psi u \;} \exp\left[-i\theta_i \frac{\sigma_i}{2} + b_i \frac{\sigma_i}{2}\right]_\alpha{}^\beta \psi_\beta$

$\left(0, \frac{1}{2}\right)$ $M \leftrightarrow N$ $\psi^{\dot\alpha}$ $J_i = \sigma/2$ $\dot\alpha$ $\dot\beta$ $\psi^{\dot\beta}$

$K_i = +i\sigma/2$

$\left(\frac{1}{2}, \frac{1}{2}\right)$

Coord: $x^\mu \to x^\mu + \Lambda^\mu{}_\nu x^\nu \approx x^\mu + \omega^\mu{}_\nu x^\nu$

States $|\psi\rangle \to U(\Lambda)|\psi\rangle \qquad U(\Lambda) = \exp\left(-\frac{i}{2\hbar}\omega_{\mu\nu} J^{\mu\nu}\right)$

Operators $\varphi_a \to M_{ab}\varphi_b \qquad M_{ab} = \delta_{ab} - \boxed{\frac{i\omega_{\mu\nu}}{2}} \mathcal{J}^{\mu\nu}_{ab}$

$[J, J] = i\epsilon J$

Right Left sum

$[\mathcal{J}, \mathcal{J}] = i\epsilon \mathcal{J} \to \left(\frac{m}{2}, \frac{n}{2}\right) \quad M \quad N$

$(1,0)$

spin-1

$M^i_{ab} = -i\epsilon_{iab}$ $\quad N^i = [0]$

jk

$J^i_{ab} = -i\epsilon_{iab}$ ← spin-1

$K^i_{ab} = -\epsilon_{iab}$ ← sorta spin 1

$$A^\mu = \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E & -E & -E \\ +E & 0 & -B_z & B_y \\ E & B_z & 0 & -B_x \\ E & -B_y & B_x & 0 \end{bmatrix}$$

antisymm

if $B_i = iE_i$ then

$$V_j \xrightarrow{\Lambda} \exp\left[-i\theta_i(-i)\epsilon_{ijk} + ib_i(-\epsilon_{ijk})\right] V_k$$

Rot: $V_j \to V_j - \epsilon_{ijk} V_k \theta_i$ ✓

Boost $V_j \to V_j + i\epsilon_{ijk} b_i V_k$

$V_j + i\vec{b} \times \vec{V}$ huh?

$$V_j \xrightarrow{\theta} V_j + \epsilon_{jik}\theta_i V_k = \vec{V} + \vec{\theta}\times\vec{V}$$

$$V_j \xrightarrow{b} V_j + i\,\vec{b}\times\vec{V}$$

$F^{\mu\nu}$ if $\vec{B} = -i\vec{B}$

Boost $\vec{E} \to \vec{E} - \vec{b}\times\vec{B}$ → $\vec{E} + i\vec{b}\times\vec{E}$

if $\vec{B} = -i\vec{E}$

$\vec{B} \to \vec{B} + \vec{b}\times\vec{E}$ → $\vec{B} + i\vec{b}\times\vec{B}$

$(0,1)$ $F^{\mu\nu}$ if $\vec{B} = +i\vec{E}$

$(1,0) \oplus (0,1) \to F^{\mu\nu}$ not A^μ

$\underbrace{\left(\frac{1}{2}, \frac{1}{2}\right)}_{\otimes}$ 4 comp. $\frac{1}{2} \otimes \frac{1}{2} = \underline{1} \oplus 0$

2-comp 2comp $\underline{\underline{\text{4-vectors}}}$

$\vec{A}$ A^0 $\begin{bmatrix} A^0 \\ \vec{A} \end{bmatrix}$

$(0,0)$ Scalar ← $\underline{\underline{\text{Fundamental}}}$ fields of QFT must be

$(\frac{1}{2}, 0)$ spin $\frac{1}{2}$ ←

$(0, \frac{1}{2})$ spin $\frac{1}{2}$ ←

$(\frac{1}{2}, \frac{1}{2})$ 4-vectors ← $\underline{\underline{\text{If}}}$ gauge fields of gauge symmetry (EM)

$\underline{\underline{A}}^\mu \to \underline{\underline{J}}^\mu$

$(1,0)$ } $F^{\mu\nu}$

$(0,1)$

$(1, \frac{1}{2})$ } → in supergravity, gravitino, 1 $\underline{\underline{\text{once}}}$

$(\frac{1}{2}, 1)$

$(1, 1)$ ← $\underline{\underline{\text{If}}}$ it only $\underline{\text{1'ce}}$ & it's the $\underline{\underline{\text{graviton}}}$ of GR

So let's understand spin-1/2 fields

$(\tfrac{1}{2},0)$ and $(0,\tfrac{1}{2})$

Met earlier

$$\psi^\alpha \xrightarrow{\Lambda} \tilde{\psi}^\alpha(\Lambda^{-1}x) = S^\alpha{}_\beta(\Lambda)\,\psi^\beta(\Lambda^{-1}x)$$

σ_2 is im

$$\Rightarrow S^\alpha{}_\beta = \exp\left[\left(-i\,\theta_i + b_i\right)\frac{\sigma_i}{2}\right]^\alpha{}_\beta$$

rot angles

boost magnitudes (rapidity)

complex matrix

ψ must be a $\mathbb{C}$ field

Then $\psi^\dagger \neq \psi$ is different from ψ.

$$\mathcal{L}(\psi,\psi^\dagger) = \mathcal{L}^\dagger(\psi,\psi^\dagger)$$

How does $\psi^\dagger$ transform?

$$(\psi^\alpha)^\dagger = \psi^{\dot\alpha}$$

$\dot\alpha$ to remind me — degree of ψ.

→ Transform rule?

$$\psi^{\dot\alpha} = (\psi^\alpha)^\dagger \longrightarrow \left(S^\alpha{}_\beta \, \psi^\beta \right)^\dagger$$

$$= \underbrace{\exp\left[(\pm i\theta_i \pm b_i) \frac{\sigma_i^{*}}{2} \right]^{\dot\alpha}}_{\text{huh?? New representation? NO}}{}_{\dot\beta} \, \psi^{\dot\beta}$$

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$(\sigma^1)^* = \sigma^1 \qquad (\sigma^3)^* = \sigma^3 \qquad (\sigma^2)^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -\sigma^2$$

$$\sigma^2\sigma^1\sigma^2 = -\sigma^2\sigma^2\sigma^1 \qquad \sigma^2\sigma^3\sigma^2 = -\sigma^2\sigma^2\sigma^3 \qquad \sigma^2\sigma^2\sigma^2 = +\sigma^2\sigma^2\sigma^2$$

$$\sigma^2\sigma^{1*}\sigma^2 = -\sigma^1 \qquad \sigma^2\sigma^{3*}\sigma^2 = -\sigma^3 \qquad \sigma^2\sigma^{2*}\sigma^2 = -\sigma^2$$

$$\sigma^2 e^{A} \sigma^2 = e^{\sigma^2 A \sigma^2} \qquad \sigma^2\sigma^2 = 1$$

$$\sigma_2\left(1 + A + \frac{A^2}{2} + \frac{A^3}{6} + \dots\right)\sigma_2 = \left(\sigma_2\sigma_2 + \sigma_2 A \sigma_2 + \frac{\sigma_2 A \sigma_2\sigma_2 A \sigma_2}{2} + \frac{\sigma_2 A \sigma_2\sigma_2 A \sigma_2\sigma_2 A \sigma_2}{6} + \dots\right)$$

$$\sigma_i^* = -\sigma_2 \sigma_i \sigma_2$$

$$e^{[(i\theta+b)\sigma^*]} \psi^{\dot\alpha} = [\sigma_2(-i\theta-b)\sigma_i \sigma_2] \psi^{\dot\alpha}$$

$$e^{(-i\theta-b)\sigma}$$

$$= \sigma_2 \underbrace{e^{\,}} \underbrace{\sigma_2 \psi^{\dot\alpha}}$$

$$\underbrace{\sigma_2 \psi^{\dot\alpha}} \longrightarrow \cancel{\sigma_2} \cancel{\sigma_2} \left[\underline{e^{(-i\theta-b)\frac{\sigma}{2}}} \underbrace{\sigma_2 \psi^{\dot\alpha}} \right]$$

$$\sigma_2 \psi^{\dot\alpha} \longrightarrow \underbrace{e^{(-i\theta-b)\frac{\sigma}{2}}}_{R\text{-field trans. rule}} \sigma_2 \psi^{\dot\alpha}$$

$(0, 1/2)$ field.

$\sigma_2 \psi^{\dagger}$ is $\underline{(0\ 1/2)}$ $\underline{\text{field}}$

What transforms?

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu = x^\mu + \omega^\mu{}_\nu x^\nu$$

$$|\psi\rangle = U(\Lambda)|\psi\rangle \qquad \hat{J}^{\mu\nu}$$

$\mathcal{O}_a$, eg,

$$\psi^\alpha \to \boxed{S^\alpha{}_\beta}\, \psi^\beta \qquad (1/2, 0)$$

$$A^\mu \to \Lambda^\mu{}_\nu A^\nu \qquad (1/2, 1/2)$$

$$\varphi \to \varphi \qquad (0,0)$$

$$\psi_{\dot\alpha} = \sigma_2 \psi^{\dot\alpha} \qquad (0, 1/2)$$

$$\left(\frac{m}{2}, \frac{n}{2}\right)$$

spin: $\frac{m}{2} \otimes \frac{n}{2}$

Lorentz - complicated

$$\mathcal{L} = \boxed{\psi^\alpha \psi_{\dot\beta}}$$