Reminder - Spinors & Lorentz Group

boost on z-axis

$$\begin{pmatrix} e^{b/2}\psi^1 \\ e^{-b/2}\psi^2 \end{pmatrix}$$

L-spinor $\psi^{\alpha} \longrightarrow S^{\alpha}{}_{\beta}\,\psi^{\beta}$

$\alpha = 1, 2$ 2 C comp.'s

$$S^{\alpha}{}_{\beta} = \exp\left[(-i\theta_i + b_i)\frac{\sigma_i}{2}\right]^{\alpha}{}_{\beta}$$

Rotations

Frame changes

R-spinor $\psi_{\dot\alpha} \longrightarrow S_{\dot\alpha}{}^{\dot\beta}\,\psi_{\dot\beta}$

$$S_{\dot\alpha}{}^{\dot\beta} = \exp\left[(-i\theta_i - b_i)\frac{\sigma_i}{2}\right]_{\dot\alpha}{}^{\dot\beta}$$

note sign

Dagger: $(\sigma_2\psi^{\alpha})^{\dagger} = \psi_{\dot\alpha}$

$\circ$ - Hermitian conj

$(\sigma_2 \psi_{\dot\alpha})^\dagger = \psi^\alpha$

$R \to L$ $\quad$ Hermitian Conj.

$L \to R$

$\mathcal{L}(\psi_1^\alpha, \psi_{2\dot\alpha}, \dots)$ Lagrangian Dens.

must be Lorentz scalars

how to combine into scalars?

2 spin-1/2 $\boxed{\psi_1^a, \psi_2^b}$

$\frac{1}{2} \otimes \frac{1}{2} \to 1 \oplus 0$

$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ is spin 0

$|\uparrow\uparrow\rangle$ etc] spin-1.

$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$?

$\uparrow$ 1'st comp

$\downarrow$ 2'nd comp

$|\uparrow\downarrow\rangle = \psi_1^1 \psi_2^2$

$-|\downarrow\uparrow\rangle = -\psi_1^2 \psi_2^1$

$$[\psi_1^1 \;\; \psi_2^2]\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \psi_1^1 \\ \psi_1^2 \end{bmatrix}$$

$$\epsilon_{ab}\psi_1^a\psi_2^b = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

spin 0 comb.

$$\epsilon_{ab} = (i\sigma_2)_{ab}$$

$$\boxed{\epsilon_{ab}\psi_1^a\psi_2^b} \xrightarrow{\Lambda} \epsilon_{ab}\left[\exp\left((-i\theta+b)_i\frac{\sigma_i}{2}\right)\right]^a{}_c \left[\exp\left((-i\theta+b)_i\frac{\sigma_i}{2}\right)\right]^b{}_d$$

$$\left(e^{(i\theta-b)\frac{\sigma}{2}}\right)^b{}_a\left(e^{(-i\theta+b)\sigma}\right)^a{}_c \;\; \epsilon_{bd}\psi_c\psi_d$$

$$\boxed{\epsilon\sigma = -\sigma^*\epsilon}$$

$$\epsilon_{ab}\left[e^{(-i\theta+b)\frac{\sigma}{2}}\right]^b{}_d = \left[e^{-(-i\theta+b)\frac{\sigma^*}{2}}\right]^b{}_a \epsilon_{bd}$$

$$= \left[e^{(i\theta-b)\frac{\sigma}{2}}\right]^b{}_a \epsilon_{bd}$$

$$\delta^b{}_c$$

$$\boxed{\epsilon_{cd}\psi_c\psi_d}$$

Will this work for $\psi^\alpha \psi_\beta$? $\epsilon^\alpha_\beta \psi^\alpha \psi_\beta$?

L   R

$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$?

Spin-0. But not boost-inv.

L-handed $\begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} \xrightarrow{bz} \begin{bmatrix} e^{b/2}\psi^1 \\ e^{-b/2}\psi^2 \end{bmatrix}$ $\otimes$ $e^{b}|\uparrow\downarrow\rangle - e^{-b}|\downarrow\uparrow\rangle$

R-handed $\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \xrightarrow{bz} \begin{bmatrix} e^{-b/2}\psi_1 \\ e^{+b/2}\psi_2 \end{bmatrix}$

t-comp of 4-vect.

2 L's $\epsilon_{ab}\psi^a\psi^b$ scalar $\epsilon_{ab}\sigma^{\ b}_{i\ b'}\psi^a_1\psi^{b'}_2$ spin-1

(1,0)

Combining one L, one R? 4 comp's.

Spin: $\frac{1}{2} \otimes \frac{1}{2} \to \underline{1 \oplus 0}$

Lorentz: $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = \underline{(\frac{1}{2}, \frac{1}{2})}$ 4-vector? $\underline{\text{yes}}$

ψ^α L-handed object $[\ ]$

$\psi_\alpha \equiv \underline{\epsilon_{\alpha\beta}} \psi^\beta$ think of $[\ \ ]$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\psi_{1\alpha} \psi_2^\alpha = \underbrace{\epsilon_{\alpha\beta} \psi_1^\beta \psi_2^\alpha}$, $[\ \ ] [\ ]$

4 indep.

$[\ \ ]$

$\underline{\psi^{\dot\alpha}} [\sigma^\mu_{\dot\alpha\alpha}] \psi^\alpha$ $[\ ]$

$\psi_{\dot\alpha}$ R-handed

$\underline{\epsilon}^{\dot\alpha\dot\beta} \psi_{\dot\alpha} \psi_{\dot\beta}$

$\psi_{\dot\alpha} [\ ]$

$\psi^{\dot\alpha} [\ \ ]$

$\sigma^\mu = \begin{bmatrix} \mathbb{1} \\ \sigma^x \\ \sigma^y \\ \sigma^z \end{bmatrix}$

$$\left(\psi^{\dot\beta}\,\underline{\sigma^{\mu}_{\dot\beta\alpha}}\,\psi^{\alpha}\right)^{T}$$

$$\sigma^{\mu}_{\dot\alpha\alpha} = \begin{bmatrix} \delta_{\dot\beta\alpha} \\ \sigma^{x}_{\dot\beta\alpha} \\ \sigma^{y}_{\dot\beta\alpha} \\ \sigma^{z}_{\dot\beta\alpha} \end{bmatrix}$$

$-\epsilon \mathbb{1} \epsilon = \mathbb{1}$

$-\epsilon \sigma \epsilon = -\sigma^{T}$

$\ldots$

Λ

$$\psi^{\dot\beta}\, \underbrace{e^{(i\theta_i + b_i)\frac{\bar\sigma_i}{2}}\, \underline{\boxed{\sigma^{\mu}}}\, e^{(-i\theta_i + b_i)\frac{\sigma_i}{2}}}\, \psi^{\alpha}$$

$$\underline{\Lambda^{\mu}{}_{\nu}\, \sigma^{\nu}}$$

$$\underline{\psi_{\alpha}}, \quad \bar\sigma^{\mu\alpha\dot\alpha}, \quad \underline{\psi_{\dot\alpha}}$$

$$\underline{\bar\sigma^{\mu}} = \begin{bmatrix} \mathbb{1} \\ -\sigma \\ -\sigma \\ -\sigma \end{bmatrix}$$

Notation: Weyl Notation Herman Weyl

Weyl Spinors ψ^{α} $\psi_{\dot\alpha}$

Combined using ϵ, σ^{μ}

ϵ Klebsch-Gordan coeff $(\frac{1}{2},0)(\frac{1}{2},0)$

$(0,0)$

Klebsch-Gordan

$R + L \to$ Lorentz index A^{μ}

Very often fields come in pairs

L - electron
R - electron
electron

L - up q
R - up q
up quark

All leptons, quarks (exc. ν's) are like this

$$\begin{bmatrix} e_{R\dot\alpha} \\ e_L{}^{\alpha} \end{bmatrix} \begin{matrix} \leftarrow 2 \text{ comp } R \\ \leftarrow 2 \text{ comp } L \end{matrix}$$

4 comp's

e_R, e_L indep. distinct fields

$$\underset{\text{4-comp}}{e} = \begin{bmatrix} e_{R\dot\alpha} \\ e_L^{\alpha} \end{bmatrix}$$

Dirac Spinor

$$\bar{e} = \begin{bmatrix} e_L^{\dot\alpha} & e_{R\alpha} \end{bmatrix}$$ daggers of

$$= e^{\dagger} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\bar{e}e = \underbrace{e_L^{\dot\alpha} e_{R\dot\alpha} + e_L^{\alpha} e_{R\alpha}}$$ ← scalar

$$\bar{e}\gamma^{\mu}e = \text{4-vector} = \underbrace{e_L^{\alpha} \bar{\sigma}^{\mu}_{\alpha\dot\alpha} e_L^{\dot\alpha}} + \underbrace{e_{R\dot\alpha} \bar{\sigma}^{\mu\dot\alpha\alpha} e_{R\alpha}}$$

$$\gamma^{\mu} \Rightarrow \gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \gamma^i = \begin{bmatrix} 0 & +\sigma_i \\ -\sigma_i & 0 \end{bmatrix}$$

note: $(\gamma^i)^{\dagger} = -\gamma^i$

$$\begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} = -\begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix}$$

$$\gamma^{\mu} = \begin{bmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{bmatrix}$$

$$\psi^\alpha \to S^\alpha{}_\beta\, \psi^\beta$$

$$\begin{bmatrix} e_{R\dot\alpha} \\ e_L^\alpha \end{bmatrix} \to \begin{bmatrix} S_{\dot\alpha}{}^{\dot\beta} & \\ & S^\alpha{}_\beta \end{bmatrix} \begin{bmatrix} e_{R\dot\beta} \\ e_L^\beta \end{bmatrix}$$

$$S = e^{-i\sum \sigma_{\alpha\beta} \frac{\omega_{\alpha\beta}}{2}} \; \cdots$$

$$\exp -\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}$$

$$\frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$\exp -\frac{i}{2}\omega_{\mu\nu} \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

γ's obey algebra: Clifford Algebra

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

1) What if I only have 2-comp field χ_α
(neutrinos - Standard Model - all fields)

$$\Psi_M = \begin{bmatrix} \chi^{\dot\alpha} \\ \chi_\alpha \end{bmatrix} \begin{array}{l} \leftarrow \text{Herm. conj.} \\ \leftarrow \text{field} \end{array}$$

Majorana notation.

Use Dirac notation $\rightarrow$
Careful: $\bar\Psi, \Psi$ rearranging something.
notation has equivalences. Careful.
(Ch 1, 2, Burgess & Moore)

2) Similarity Transform.

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \longrightarrow S \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

eg, $\begin{bmatrix} \uparrow \\ \uparrow \\ \downarrow \\ \downarrow \end{bmatrix}$

$\Psi \rightarrow S\Psi$ Basis change.

$\bar\Psi \rightarrow \bar\Psi S^{-1}$

$\gamma^\mu \rightarrow S\gamma^\mu S^{-1}$

$\bar\Psi\gamma^\mu\Psi \rightarrow \bar\Psi \cancel{S^{-1}S}\gamma^\mu \cancel{S^{-1}S}\Psi$

$\bar\Psi\Psi \rightarrow \bar\Psi \cancel{S^{-1}S}\Psi$

We chose Chiral Basis - useful for very relativistic part.

Dirac used another basis - useful for $v \ll c$

γ^μ not same : $\gamma^\mu \to \underline{\underline{S \gamma^\mu S^{-1}}}$

Clifford Algebra $\underline{\underline{\{\gamma^\mu, \gamma^\nu\}}} \to S\gamma^\mu \cancel{S^{-1} S} \gamma^\nu S^{-1}$

$+ S\gamma^\nu \cancel{S^{-1} S} \gamma^\mu S^{-1}$

$= S\underbrace{(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)}_{2g^{\mu\nu}\mathbb{1}} S^{-1}$

$= 2g^{\mu\nu} \underbrace{S \mathbb{1} S^{-1}}_{1} = \underline{\underline{2g^{\mu\nu}}}$

2, 3 dim : 2×2 γ^μ

4, 5 dim : 4×4 γ^μ

6, 7 dim : 8×8 γ^μ

8, 9 16×16 γ^μ

⋮

$$\mathcal{L}(\psi) = \underbrace{-m\bar{\hat{\psi}}\hat{\psi}}_{\text{scalar!!}} + \underbrace{i\bar{\hat{\psi}}\gamma^\mu\partial_\mu\hat{\psi}}_{\text{scalar}} + \cancel{\bar{\psi}\partial^\mu\partial_\mu\psi}$$

small effects....

$$\mathcal{L}^\dagger = \mathcal{L}$$

$$\psi^* = \psi^\dagger \qquad \underbrace{\bar{\psi} = \psi^\dagger\gamma^0}_{\text{in our basis}}$$

$$(\psi_1^\dagger\gamma^0\psi_2)^\dagger = \psi_2\gamma^{0\dagger}\psi_1$$

$$= \underbrace{\psi_2^\dagger\gamma^0}\psi_1$$

$$\bar{\psi}_2\psi_1$$

$\gamma^{0\dagger}$ is what?

$$= \gamma^0$$

$$(\bar{\psi}_1\psi_2)^\dagger = \bar{\psi}_2\psi_1$$

$$(\bar{\psi}\psi)^\dagger = \bar{\psi}\psi$$

What about $\bar\psi \gamma^\mu \partial_\mu \psi$??

$$(\bar\psi \gamma^\mu \partial_\mu \psi)^\dagger = \partial_\mu \psi^\dagger \gamma^{\mu\dagger} \gamma^0 \psi$$

$\psi^\dagger \gamma^0$

$\gamma^{\mu\dagger} \gamma^0 = \gamma^0 \gamma^\mu$ not obvious. Why?

if $\mu = 0$: $\rightarrow$ $\gamma^{0\dagger} \gamma^0 = \gamma^0 \gamma^0$

if $\mu = i$: $\gamma^{i\dagger} \gamma^0 = -\gamma^i \gamma^0 = +\gamma^0 \gamma^i$

$\gamma^0 \gamma^i + \gamma^i \gamma^0 = 2g^{i0} = 0$

$\gamma^0 \gamma^i = -\gamma^i \gamma^0$

$$= \partial_\mu \psi^\dagger \gamma^0 \gamma^\mu \psi = (\partial_\mu \bar\psi) \gamma^\mu \psi$$

That's OK

$$\mathcal{L} = -m\bar\psi\psi + i\bar\psi\gamma^\mu\partial_\mu\psi - \frac{i}{2}\partial_\mu\left[\bar\psi\gamma^\mu\psi\right]$$

total deriv

$$\frac{i}{2}\left(\bar\psi\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar\psi)\gamma^\mu\psi\right)$$

dagger is $\partial_\mu\bar\psi\gamma^\mu\psi$ dagger is $\bar\psi\gamma^\mu\partial_\mu\psi$

combin. flips sign under †

$$A\overleftrightarrow{\partial_\mu}B = \frac{1}{2}\left(-\partial_\mu A\,B + A\partial_\mu B\right)$$

Renorm.

$$\mathcal{L} = -m\bar\psi\psi + i\bar\psi\gamma^\mu\overleftrightarrow{\partial_\mu}\psi + \cancel{\frac{\lambda}{2}\bar\psi\psi\bar\psi\psi}$$

$\mathcal{L}$: E^4; m: E; $\bar\psi\psi$: E^3; $\bar\psi$: $E^{3/2}$; ∂_μ: E^1; ψ: $E^{3/2}$

λ: E^{-2}; $\bar\psi\psi\bar\psi\psi$: $E^{3/2}E^{3/2}E^{3/2}E^{3/2}$ = E^6

E^{-2} = (fund. scale)$^{-2}$

$$\mathcal{L}(\varphi, \psi, A_\mu) = [\mathcal{L}_{\varphi, A^\mu}] \boxed{-m\bar\psi\psi + i\bar\psi\gamma^\mu\partial_\mu\psi}$$

$$+g\,\varphi\bar\psi\psi \quad + e\bar\psi\gamma^\mu\psi A_\mu$$

int. effs.

Yukawa Coupling

gauge coupling

free spinor Field. Can we solve that?

Dirac Eq.

Fields obey anti-comm. relations.

Fermions

$$(\gamma^0)^\dagger \rightarrow (S\gamma^0 S^{-1})^\dagger = S^{-1\dagger}\gamma^{0\dagger}S^\dagger = S^{-1\dagger}\gamma^0 S^\dagger$$

$$= S\gamma^0 S^{-1}$$