Lecture 17. "Classical" Dirac Equation

Last time: 3 field types

Scalars $\varphi(x) \xrightarrow[\omega \text{ infinites.}]{x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu} \varphi(\Lambda^{-1} x)$ stays same

4 Vectors $A^\mu(x) \xrightarrow{x^\mu \to x + \omega x} A^\mu + \omega^\mu{}_\nu A^\nu(\Lambda^{-1} x)$ same as x^μ

Spinor $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} \psi_\alpha \\ \hline \psi^\alpha \end{bmatrix} \longrightarrow \underline{\underline{S}}\,\psi$ spin-1/2

$$S = \mathbb{1} + \frac{1}{2}\omega_{\mu\nu} S^{\mu\nu} \qquad S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Building Tensors

$\varphi \quad \partial_\mu \varphi \quad \partial_\mu \partial_\nu \varphi \quad \ldots$

$A^\mu \quad \partial^\nu A^\mu \quad \partial_\mu A^\mu \quad \ldots$

scalar

Integer Spin

$\bar{\psi} \psi \rightarrow$ scalar

$\bar{\psi} \gamma^\mu \psi \rightarrow$ 4 vector

$\rightarrow \bar{\psi} [\gamma^\mu, \gamma^\nu] \psi \rightarrow$ Antisymm Tensor

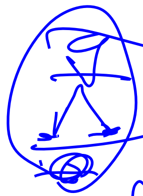
$\bar{\psi} [\gamma^\mu, [\gamma^\nu, \gamma^\alpha]] \psi = \epsilon^{\mu\nu\alpha\beta} (\ )_\beta$ pseudovector

$\bar{\psi} [\gamma^\mu, [\gamma^\nu, [\gamma^\alpha, \gamma^\beta]]] \psi = \epsilon^{\mu\nu\alpha\beta} (\ )$ pseudo scalar

$$\mathcal{L}(\bar\psi, \psi, \varphi, A_\mu) = \underbrace{\bar\psi(-m + i\gamma^\mu \partial_\mu)\psi}_{\text{free particle}} + \boxed{g\varphi\bar\psi\psi + ie\bar\psi\gamma^\mu A_\mu \psi}$$

Classical level - what does $\mathcal{L}$ predict??

Class. Field Eq. of $\psi(x^\mu)$



$$\cancel{F_{\mu\nu}\,\bar\psi[\gamma^\mu,\gamma^\nu]\psi}$$

Dim $2 + 3/2 + 3/2 = 5$

$$S = \int d^4y\, \mathcal{L}$$

$$0 = \frac{\delta S}{\delta \bar\psi(x)} \Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar\psi} = \frac{\delta \mathcal{L}}{\delta \bar\psi} = 0 = [-m + i\gamma^\mu \partial_\mu]\psi$$

$\partial_\mu \dfrac{\partial \mathcal{L}}{\partial \partial_\mu \bar\psi} = 0$

Dirac Equation

$$\frac{\delta S}{\delta \psi} \to \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = \frac{\partial \mathcal{L}}{\partial \psi} \to [\quad] \partial\bar\psi = 0$$

$$(-m + i\gamma^\mu \partial_\mu)\,\psi(x) = 0, \qquad \text{Class. sol'ns of spinor QFT}$$

$$(x+1)(x-1) = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$\boxed{x = \pm 1}$$

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\left[-\gamma^\nu \partial_\nu \gamma^\mu \partial_\mu + \cancel{m i \gamma^\mu \partial_\mu} - \cancel{i\gamma^\mu \partial_\mu m} - m^2\right]\psi(x) = 0$$

$$AB = \tfrac{1}{2}(AB+BA) + \tfrac{1}{2}(AB-BA)$$

$$= \tfrac{1}{2}\{A,B\} + \tfrac{1}{2}[A,B]$$

$$-\gamma^\nu\gamma^\mu \, \partial_\nu\partial_\mu$$

$$-\tfrac{1}{2}\underbrace{\{\gamma^\nu,\gamma^\mu\}}_{= 2g^{\mu\nu}}\,\frac{\partial_\nu\partial_\mu + \partial_\mu\partial_\nu}{2}$$

sol'n of obeys. Not vice versa.

$$\boxed{\left[-\partial_\mu\partial^\mu - m^2\right]\psi = 0}$$

Klein Gordon Eq.

Sol'ns of form $\psi(x) = \psi e^{-ip_\mu x^\mu}$

$$(+p_\mu p^\mu - m^2)\psi = 0$$

$$\boxed{p^2 = m^2}$$

$$E^2 = \vec{p}^{\,2} + m^2$$

Sol'ns of ψ if Classical EM field present.

B - field

$\vec{E}$ - field of form $\frac{e\,\hat{r}}{4\pi r^2}$ Coulomb $\vec{E}$ field

$$\mathcal{L}_{Dirac} = \bar{\psi}\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi$$ Minimal coupling

Dirac Eq. $\left[i\gamma^\mu\underbrace{(\partial_\mu - ieA_\mu)}_{D_\mu} - m\right]\psi = 0$

D_μ Covariant Deriv.

$[D_\mu, D_\nu] \neq 0$

$$(i\gamma^\nu D_\nu + m)(i\gamma^\mu D_\mu - m)\psi = 0$$

$$\left[-\gamma^\nu\gamma^\mu D_\nu D_\mu - m^2\right]\psi = 0$$

$$\left[-\gamma^\nu\gamma^\mu D_\nu D_\mu - m^2\right]\psi = 0$$

$$\gamma^\nu\gamma^\mu = \frac{1}{2}\left(\{\gamma^\nu,\gamma^\mu\} + [\gamma^\nu,\gamma^\mu]\right)$$

$$D_\nu D_\mu = \frac{1}{2}\left(\{D_\nu, D_\mu\} + [D_\nu, D_\mu]\right)$$

$$\left(-\frac{1}{4}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2g^{\mu\nu}}\{D_\nu, D_\mu\} - \frac{1}{4}[\gamma^\mu,\gamma^\nu][D_\nu, D_\mu] - m^2\right)\psi = 0$$

$$-\frac{1}{2}\{\gamma^\mu, D_\mu\} = -D^2$$

$[D_\nu, D_\mu]$ = what? $-ieF_{\nu\mu}$

$$-\frac{1}{4}[\gamma^\mu,\gamma^\nu] = -i S^{\mu\nu}$$ strength of frame changes

$$S^{0i} = \begin{bmatrix} -\frac{\sigma_i}{2} & 0 \\ 0 & +\frac{\sigma_i}{2} \end{bmatrix}$$

$$S^{ij} = -i\epsilon^{ijk}\begin{bmatrix} \frac{\sigma_k}{2} & 0 \\ 0 & \frac{\sigma_k}{2} \end{bmatrix} - S_k$$

$$\boxed{[D_\nu, D_\mu]}\chi = -ieF_{\nu\mu}\chi$$

$$(\partial_\nu - ieA_\nu)(\partial_\mu - ieA_\mu)\chi - (\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu)\chi$$

$$[\partial_\nu\partial_\mu - ieA_\nu\partial_\mu - ie\,\partial_\nu A_\mu - e^2 A_\nu A_\mu]\chi$$

$$\partial_\nu[A_\mu\chi] = (\partial_\nu A_\mu)\chi + A_\mu\partial_\nu\chi$$

$$\cancel{\partial_\nu\partial_\mu} - \cancel{ieA_\nu\partial_\mu} - ie(\partial_\nu A_\mu) - \cancel{ieA_\mu\partial_\nu} - \cancel{e^2A_\nu A_\mu}$$

$$-\cancel{\partial_\mu\partial_\nu} + \cancel{ieA_\mu\partial_\nu} + ie(\partial_\mu A_\nu) + \cancel{ieA_\nu A_\mu} + \cancel{e^2A_\mu A_\nu}$$

$$\chi$$

$$-ie(\partial_\nu A_\mu - \partial_\mu A_\nu) = -ieF_{\nu\mu}\chi$$

$$\left[-D_\mu D^\mu - \left(-i S^{\mu\nu}\right)\left(-ie F_{\mu\nu}\right) - m^2\right]\psi = 0$$

Scalar: $\left(-D_\mu D^\mu - m^2\right)\varphi = 0$

$$S^{\mu\nu} \quad F_{\mu\nu}$$

$$12 \cdot 12$$

$$21 \cdot 21$$

$F_{\mu\nu} = F_{ij}$ B-field ... $-2\vec{S}\cdot\vec{B}$

$$\psi = e^{-ip\cdot x}\psi_0$$

$$D_\mu D^\mu \to -p^2 = -E^2 + \vec{p}^2$$

$$E^2 - \vec{p}^2 - 2e\vec{S}\cdot\vec{B} - m^2 = 0$$

$$E = \sqrt{p^2 + m^2 + 2e\vec{S}\cdot\vec{B}}$$

$$= \sqrt{p^2+m^2} + \frac{eS\cdot B}{\sqrt{p^2+m^2}}$$

$$E = m + \frac{eS\cdot B}{m}$$

Ball of massive, charged stuff

$\mu =$

$$j = \frac{e\omega}{2\pi}$$

$$A = \pi r^2$$

$$\mu = \frac{e\omega r^3}{2}$$

$$S = m\omega r^2 \qquad \mu = \frac{e}{2m} S$$

$$\frac{\mu}{S} = \frac{e\omega r^2/2}{m\omega r^2} = \frac{e}{2m}$$

$\frac{\text{Mag moment}}{\text{Ang mom}}$ $\frac{\text{normal}}{}$ spinning charged thing

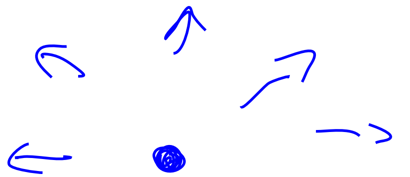
$\mu \cdot B$ $\underline{\text{Mag energy}} = \boxed{\frac{e}{2m} S \cdot B}$

$\underline{\text{Actual}}$ $\frac{e}{m} S \cdot B$

$$= g \frac{e}{2m} \vec{S} \cdot \vec{B}$$

$$g = 2$$

Coulomb Potential $A^0 = \frac{eZ}{4\pi r}$ Z - chg nucleus 1-H ⋮



+e charge at rest

p or chg-Z nucleus

$\vec{E} = \frac{eZ}{4\pi r^3}\vec{r}$ $\qquad \underline{\vec{A} = 0}$

S.F

Dirac. $\left(-\underbrace{(\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu)}_{D^2} - m^2 + ie\begin{bmatrix} \vec{B}\cdot\vec{\sigma} & 0 \\ 0 & -\vec{E}\cdot\vec{\sigma} \end{bmatrix}\right)\psi = 0$

Energy states? $\psi(\vec{x},t) = \psi(\vec{x})e^{-i\omega t}$

$\partial_0 \to -i\omega$

$\partial_0 - ieA_0 \to -i(\omega + eA_0) = -i\left(\omega + \frac{\alpha Z}{r}\right)$

$\partial_i - ieA_i = \partial_i$

$\left[\left(\omega + \frac{\alpha Z}{r}\right)^2 + \nabla^2 - m^2 + \begin{bmatrix} i\vec{\sigma}\cdot\hat{r} & 0 \\ 0 & -i\vec{\sigma}\cdot\hat{r} \end{bmatrix}\frac{Z\alpha}{r^2}\right]\psi = 0$

Direc. Dependent

$$\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r - \frac{1}{r^2}\vec{L}^2$$

$\vec{L}$ orbital ang. mom op.

$$L^2 \Omega_{\ell m}(\theta,\varphi) = \ell(\ell+1)\Omega_{\ell m}(\theta,\varphi)$$

$$\underbrace{(\omega^2 - m^2)}_{E^2 - m^2}\psi = \Big[\underbrace{-\partial_r^2 - \frac{2}{r}\partial_r + \Big[L^2 - Z^2\alpha^2 + \big[-i\vec{\sigma}\cdot\frac{\vec{r}}{r} + i\vec{\sigma}\cdot\frac{\vec{r}}{r}\big]Z\alpha\Big]\frac{1}{r^2} - \frac{2Z\alpha\omega}{r}}_{H}\Big]\psi$$

Assume $\psi = R(r)F(\theta,\varphi)$ But $F(\theta,\varphi)$ Maybe []

$[L, H] \neq 0$ But $[(\vec{L} + \frac{\vec{\sigma}}{2}), H] = 0$

$\vec{J}$ total ang. mom.

$\rightarrow$ Seek EV's of $\vec{J}^2, J_z$

$[L^2, L + \frac{\sigma}{2}] = 0$

EV of $J^2 = j(j+1)$ $j = \frac{1}{2}, \frac{3}{2}, \ldots$

Careful: $\vec{J}$, not L, cons.

j val – mix of $l = j + 1/2 = l_+$, $l = j - 1/2 = l_-$

Sol mix

1s – $l=0$

2s – 2p mix

$\vec{\sigma}\cdot\vec{r}$ does what?

$$\langle l_+ | \vec{\sigma}\cdot\hat{r} | l_+ \rangle = 0 = \langle l_- | \vec{\sigma}\cdot\hat{r} | l_- \rangle = 0$$

$$|\vec{\sigma}\cdot\hat{r}|^2 = \mathbb{1} \qquad \langle l_+ | \vec{\sigma}\cdot\hat{r} | l_- \rangle = \langle l_- | \vec{\sigma}\cdot\hat{r} | l_+ \rangle = \pm 1$$

$$(l_+ \quad l_-)\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}\begin{pmatrix} l_+ \\ l_- \end{pmatrix}$$

and $L^2 - Z^2\alpha^2 - iZ\alpha\,\vec{\sigma}\cdot\hat{r}$

$$= \begin{bmatrix} (j+1/2)(j+3/2) - Z^2\alpha^2 & \mp iZ\alpha \\ \mp iZ\alpha & (j-1/2)(j+1/2) - Z^2\alpha^2 \end{bmatrix}$$

Eigenvalues of

$$\begin{bmatrix} (j+\tfrac{1}{2})(j+\tfrac{3}{2}) - Z^2\alpha^2 & \mp iZ\alpha \\ \mp iZ\alpha & (j-\tfrac{1}{2})(j+\tfrac{1}{2}) - Z^2\alpha^2 \end{bmatrix}$$??

$$\lambda_1 = \sqrt{(j+\tfrac{1}{2})^2 - Z^2\alpha^2}\left(\sqrt{} + 1\right)$$

$$\lambda_2 = \sqrt{}\left(\sqrt{} - 1\right)$$

EV's for $\left(\omega^2 - Z^2\alpha^2\left[\right]\right)\frac{1}{r^2}$

part

Radial Problem

$$(\omega^2 - m^2)R(r) = \left[-\partial_r^2 - \frac{2}{r}\partial_r + \frac{\lambda_1 \text{ or } \lambda_2}{r^2} - \frac{2Z\alpha\omega}{r}\right]R(r)$$

Schrö

$\ell(\ell+1)$ not $\sqrt{}\left(\sqrt{}+1\right)$

Principal EV n,

$$j = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, n - \tfrac{1}{2}$$

Energy??

$$\underbrace{\delta_j = j+\tfrac{1}{2} - \sqrt{(j+\tfrac{1}{2})^2 - Z^2\alpha^2}}_{} \simeq \underbrace{\frac{Z^2\alpha^2}{2j+1}}_{} + \mathcal{O}(Z^6\alpha^6)$$

wrong

$$\omega_{nj} = \frac{m}{\sqrt{1+\frac{Z^2\alpha^2}{(n-\delta_j)^2}}} \simeq m - \frac{mZ^2\alpha^2}{2n^2} - \frac{mZ^4\alpha^4}{n^3(2j+1)} + \frac{3}{8}\frac{mZ^4\alpha^4}{n^4}$$

Correct fine struct.

Fine struct. for scalar e^-

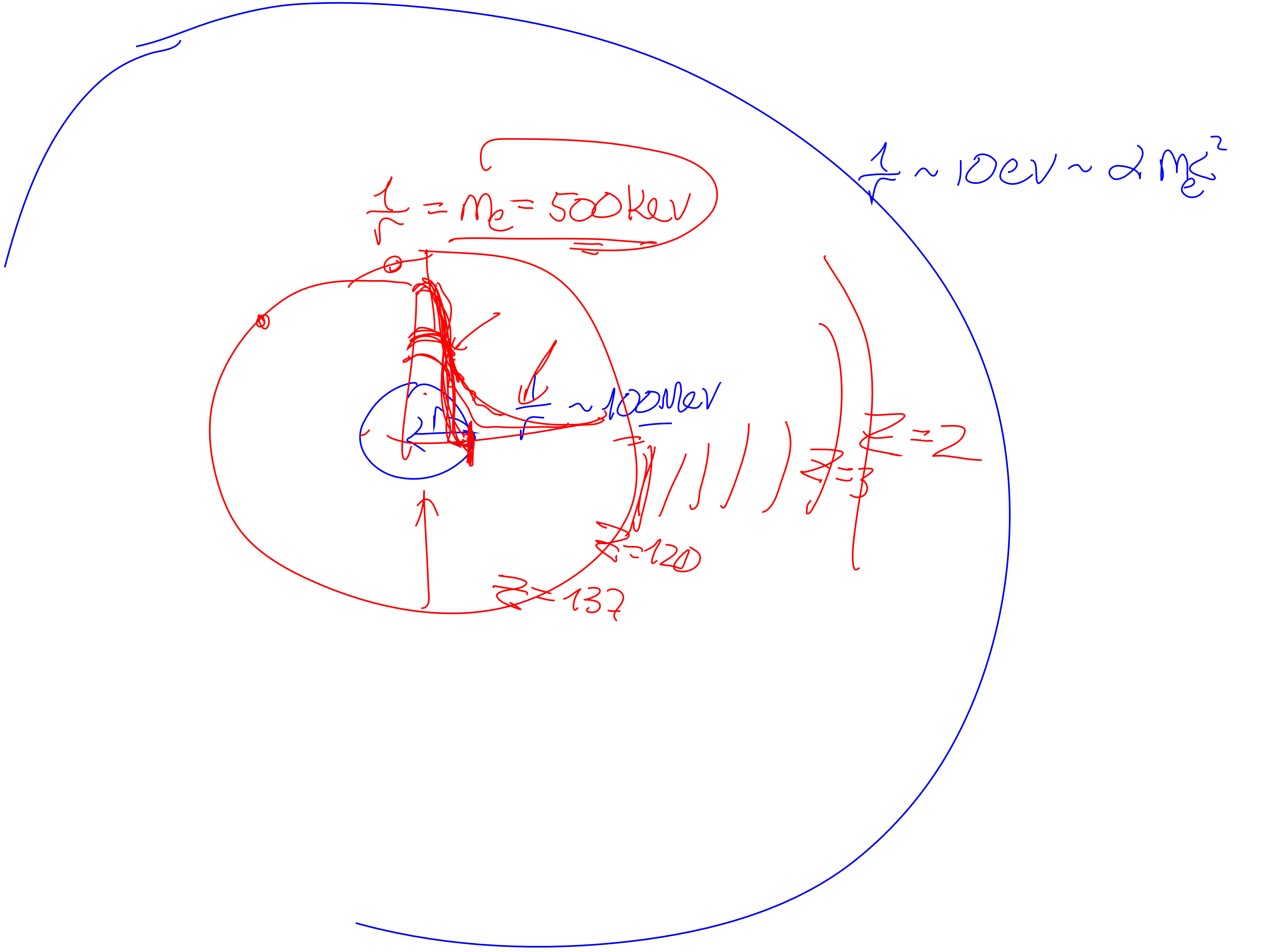
Dirac in H-like atom

Radial wave func? – could do it.

Large Z? $\underbrace{Z\alpha \longrightarrow 1,\ \delta_j = 1\quad n-\delta_j \longrightarrow 0}$ $\quad E \to 0$

$Z = \frac{1}{\alpha} = 137$. we're only up to 118.

EM field – classical. Really, it has fluct. $\longrightarrow \alpha^6$ hyperfine


1/r = m_e = 500 keV
1/r ~ 100 MeV
1/r ~ 10 eV ~ α m_e^2
Z = 2
Z = 3
Z = 120
Z = 137