Lecture 18: Fermions & Free Field Theory

1) Solutions to Dirac Equation

2) First attempt to solve thy. of free spin-½ field

$$\mathcal{L}(\bar\psi, \psi) = i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi$$

$\bar\psi = \psi^\dagger\gamma^0$

ψ Dim 3/2 $\quad \bar\psi\psi\bar\psi\psi$ etc Dim > 4

coeff in $\mathcal{L}$ has λ' Drop them

Dirac Eq $\dfrac{\delta\mathcal{L}}{\delta\bar\psi} = 0 \Rightarrow [i\gamma^\mu\partial_\mu - m]\psi$

If I multiply by $\to [i\gamma^\nu\partial_\nu + m][i\gamma^\mu\partial_\mu - m]\psi = 0$ true

$\Rightarrow [-\partial_\mu\partial^\mu - m^2]\psi = 0$ Klein-Gordon

Plane-wave Sol'ns $\to \boxed{\psi = \psi_0 e^{-ip_\mu x^\mu}}$

Most gen. sol'n to $[i\gamma^\mu \partial_\mu - m]\psi = 0$

$$\left[\psi(x,t) = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip_\mu x^\mu} \left(u(p) \text{ or } v(p) \right)\right]$$

const col vect

$$p^2 = m^2 \rightarrow (p_0)^2 - \vec{p}^2 = m^2 \qquad p_0^2 = \vec{p}^2 + m^2$$

$\rightarrow p_0 = +\sqrt{\vec{p}^2 + m^2} \rightarrow$ Column vector $u(p)\, e^{-i(\,)t}$

crazy?? well, it is a sol'n!! $(p_0 = E)$

or $\rightarrow p_0 = -\sqrt{\vec{p}^2 + m^2} \rightarrow$ column vector $v(p)\, e^{+i(\,)t}$

Dirac: $(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0$

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0}\, e^{-ip_\mu x^\mu}\,[u(p) \text{ or } v(p)]$$

∂_μ acts on x^μ $\quad \partial_\mu \to -ip_\mu$

$$\int \frac{d^3p}{(2\pi)^3 2p^0}\,\underbrace{(p^\mu\gamma_\mu - m)(u(p) \text{ or } v(p))}=0$$

Dirac Eq. in p-space

$$-[p^\mu\gamma_\mu - m]\,u(p) = 0 \qquad p^0 = +\sqrt{\vec{p}^{\,2}+m^2}$$

$$-[p^\mu\gamma_\mu - m]\,v(p) = 0 \qquad p^0 = -\sqrt{\vec{p}^{\,2}+m^2}$$

$$\psi(x) = \int_p \left[C_1^{(p)}u(p) + C_2^{(p)}v(p)\right]$$

$u(p)$, $v(p)$ standard-norm. sol'ns.

$$[p^\mu \gamma_\mu - m]\, u(p) = 0$$

$$[p^0 \gamma_0 - \vec{p} \cdot \vec{\gamma} - m\mathbb{1}]\, u(p) = 0$$

$$\gamma^3 = \begin{bmatrix} & \sigma^3 \\ -\sigma^3 & \end{bmatrix} = \begin{bmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{bmatrix}$$

$$\gamma^1 = \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -m & 0 & p^0 - p_z & -p_x + ip_y \\ 0 & -m & -p_x - ip_y & p^0 + p_z \\ p^0 + p_z & +p_x - ip_y & -m & 0 \\ +p_x + ip_y & p^0 - p_z & 0 & -m \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 0$$

$$\mathrm{Det} = (p^2 - m^2)^2$$

2 solutions for $u(p)$

ugly. so let's cheat!

Consider $\vec{p}=0$ $\quad p^0 = m \quad p^0 = -m$

$$[p^\mu \partial_\mu - m]u = 0$$

$$m\begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 0 = \begin{bmatrix} -u_1 \mp u_3 \\ -u_2 \mp u_4 \\ -u_1 - u_3 \\ -u_2 - u_4 \end{bmatrix}$$

$u_3 = u_1 \quad u_3 = -u_1$

$u_2 = u_4 \quad u_4 = -u_2$

Reason: ψ is dim $-3/2$ not 1

solutions: $\sqrt{m}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\sqrt{m}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad u_+, u_-$

$\sqrt{m}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ and $\sqrt{m}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad v_+, v_-$

Moving solutions? Use Lorentz $(v = v/c)$

$$\psi(x) = e^{-ip_0 x^0} u_+(0) \text{ or } u_-(0) \text{ is sol'n.}$$

$S(\Lambda)\,\psi(\Lambda^{-1}x)$ is also sol'n?

Boost in z-direction: $(p^0, p_z) = (\gamma m, \gamma v m)$

$$\begin{cases} \tanh b_z = v \\ \cosh b_z = \gamma \\ \sinh b_z = \gamma v \end{cases}$$

$$S(\Lambda) = \exp\begin{pmatrix} \vec{b}\cdot\frac{\vec{\sigma}}{2} & 0 \\ 0 & -\vec{b}\cdot\frac{\vec{\sigma}}{2} \end{pmatrix}$$

$$= \exp\begin{bmatrix} e^{b/2} & 0 & 0 & 0 \\ 0 & e^{-b/2} & 0 & 0 \\ 0 & 0 & e^{-b/2} & 0 \\ 0 & 0 & 0 & e^{b/2} \end{bmatrix}$$

$$e^{b/2} = ?? = \sqrt{\gamma}\sqrt{1+v}$$

$$e^{b} = \gamma(1+v) = \cosh + \sinh$$

$$u(p^0, p_z) = \sqrt{m}\begin{pmatrix} \sqrt{\gamma(1+v)} & & & \\ & \sqrt{\gamma(1-v)} & & \\ & & \sqrt{\gamma(1-v)} & \\ & & & \sqrt{\gamma(1+v)} \end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \text{ or } \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

γ large

$E \simeq p_z$

$$= \begin{pmatrix} \sqrt{\gamma(1+v)m} \\ 0 \\ \sqrt{\gamma(1-v)m} \\ 0 \end{pmatrix} = u_+ = \begin{pmatrix} \sqrt{E+p_z} \\ 0 \\ \sqrt{E-p_z} \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \sqrt{2E} \\ 0 \\ 0 \\ 0 \end{pmatrix} R \qquad \psi_R$$

spin $\parallel$ motion

$$u_- = \begin{pmatrix} 0 \\ \sqrt{E-p_z} \\ 0 \\ \sqrt{E+p_z} \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2E} \end{pmatrix} \; L \qquad \psi_L$$

spin $\uparrow\downarrow$ motion

V's, E "wrong" sign,

$$V_+ = \sqrt{\gamma m}\begin{bmatrix}\sqrt{1-\beta}\\ 0\\ \sqrt{1+\beta}\\ 0\end{bmatrix} =$$

← small

← big L-handed

spin // motion

V_- Right-handed

upper. spin ↑↓ motion

All solutions with $\vec{p}$ in z-direction.

Other direc's for $\vec{p}$?? ↪ Rotate them

$$\begin{bmatrix}1\\0\end{bmatrix}_z \longrightarrow \begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}_{x\text{-direc.}} \quad \begin{bmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{bmatrix}_{y\text{-dir.}} \quad \ldots$$

$\begin{bmatrix}1\\0\end{bmatrix}$ → spin-aligned-w-$\vec{p}$

$\begin{bmatrix}0\\1\end{bmatrix}$ → spin-opp-$\vec{p}$ 2-comp spinor.

$S(Rot)\,U(E, P_z)$

$$\Rightarrow \begin{bmatrix} e^{i\sigma\theta/2} & 0\\ 0 & e^{i\sigma\cdot\theta/2}\end{bmatrix}$$

2 solutions $\left[u_{\pm}(p) \right.$ for $E>0$

2 sol'ns $v_{\pm}(p)$ for $E<0$

s or σ label for +/− or +1/2 / −1/2

$\bar{u}_- u_- = \bar{u}_+ u_+(p) = 2m$ scalar

$\bar{u}_+ u_-(p) = 0 = \bar{u}_- u_+$

$\bar{u}_s u_{s'}(p) = 2m\, \delta_{ss'}$

$u_+^\dagger u_+(p) = 2E$ E-comp of 4-vector.

$\left[\bar{u}_s(p) \gamma^\mu u_{s'}(p) = 2p^\mu \delta_{ss'} \right.$

if p same in $\bar{u}(p)$, $u(p)$

mix p's — doesn't work.

$$\sum_s u_s(p) \bar{u}_s(p) = \gamma^\mu p_\mu + m\mathbb{1}$$

Completeness

$\vec{p}=0: \; ((\gamma^0 + \mathbb{1})m)$

$$\sum_s v_s(p) \bar{v}_s(p) = \gamma^\mu p_\mu - m\mathbb{1}$$

$(\gamma^0 - \mathbb{1})m$

Some people are too lazy $p^\mu \gamma_\mu$ $A^\mu \gamma_\mu$ or ....

Slash notation $\gamma^\mu p_\mu = \not{p}$

$$\gamma^\mu A_\mu = \not{A} = A^\mu \gamma_\mu = A^\nu \gamma_\nu = A_\nu \gamma^\nu$$

Solving Dirac Eq

$u(p)$

$v(p)$

$$\hat{\psi} = \int \frac{d^3p}{(2\pi)^3 2p_0} \left(\hat{b}\, u + \hat{d}^\dagger v \right)$$

Quantum Thy?

Scalar: $\varphi(x) \longrightarrow \hat{\varphi}(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left(\hat{a}_p e^{ip_\mu x^\mu} + \hat{a}_p^\dagger e^{-ip_\mu x^\mu} \right)$

$\mathcal{H}$-space: $\mathbb{C}$ functions over class config sp. of $\varphi(x)$

$\frac{\delta \mathcal{L}}{\delta \dot{\varphi}} = \pi$ Comm Rel. $[\varphi(x), \pi(y)] = i\delta^3(x-y) \ldots$

$\hat{\varphi}, \hat{\pi}$ def, $[\ ]$ Rel into $\underline{\underline{H}} \longrightarrow H = \int_p \omega_p \frac{(a a^\dagger + a^\dagger a)}{2}$

Same Approach here??

Guess $\mathcal{H} = \{ \mathbb{C}$ functions over class. values of $\psi(x)$

$\hat{p}_\mu$ canon 4-mom. $\frac{\delta \mathcal{L}}{\delta \partial^\mu \psi} = i \bar{\psi} \gamma_\mu$ $\qquad \hat{p}_0 = \hat{\pi} = i\bar{\psi}\gamma_0 = i\psi^\dagger$ ....

Scalar $[\pi_a(\vec{x}), \varphi_b(\vec{y})] = -i\delta^3(\vec{x}-\vec{y})\delta_{ab}$ $\quad \pi_a = -i\frac{\delta}{\delta\varphi_a}$

Spinor $\pi = i\psi^+$

$$-i[i\psi_a^+, \psi_b] = (-i)^2\delta^3(\vec{x}-\vec{y})\delta_{ab}$$

$$\boxed{[\psi_a^+(x), \psi_b(y)] = -\delta^3(x-y)\delta_{ab}}$$

$$[\psi_a, \psi_b] = 0 = [\psi_a^+, \psi_b^+]$$ sounds reasonable

What is H? $\quad H = p\dot{q} - L \qquad 2(\gamma^0)^2 = \{\gamma^0, \gamma^0\} = 2g^{00} = 2$

$$p = i\psi^\dagger = i\psi^\dagger \underbrace{\gamma^0\gamma^0} = i\bar{\psi}\gamma^0$$

$$\dot{q} = \partial_0\psi$$

$$p\dot{q} = i\bar{\psi}\underline{\gamma^0}\partial_0\psi - i\bar{\psi}\underline{\gamma^\mu}\partial_\mu\psi + m\bar{\psi}\psi$$

$$= \cancel{i\bar{\psi}\gamma^0\partial_0\psi} \cancel{- i\bar{\psi}\gamma^0\partial_0\psi} \underbrace{+ i\bar{\psi}\vec{\gamma}\vec{\partial}\psi + m\bar{\psi}\psi}$$

$$H = i\bar{\psi}\underbrace{(\vec{\gamma}\cdot\vec{\partial} + m)}\psi \qquad \text{no } \partial_0$$

Things w. $\vec{\partial}$ easiest in Fourier-land $\leftarrow$ 4 ops $\quad$ Poor basis choice

$$\hat{\psi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{+i\vec{p}\cdot\vec{x}} \begin{bmatrix} \hat{\psi}_1(p) \\ \hat{\psi}_2(p) \\ \hat{\psi}_3(p) \\ \hat{\psi}_4(p) \end{bmatrix}$$

$$\left[i\partial_{\mu} \gamma^{\mu} + m \right] \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4(p) \end{bmatrix} \rightarrow u's \quad v's$$

$$\sum_{s} \hat{b}_s(p)\, u_s(p) + \hat{d}^{\dagger}_s(p)\, v_s(p)$$

choose to renorm

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3\, 2\sqrt{\vec{p}^{\,2}+m^2}}\, e^{i\vec{p}\cdot\vec{x}} \left[\sum_{s=1} \hat{b}(p,s)\, u(p,s) + \hat{d}^{\dagger}(-p,s)\, v(-p,s) \right]$$

$$\psi^{\dagger} = \quad e^{-ip\cdot x} \left[b^{\dagger} \; u^{\dagger} \quad d \quad v^{\dagger} \right]$$

E_p

$$H = \int d^3x\, \bar{\psi}\left(i\vec{\gamma}\cdot\vec{\partial} + m\right)\psi$$

$b^\dagger \quad d$

$\sum b \, d^\dagger$

$[\psi, \psi] = 0$

$[\psi^\dagger, \psi] = -\delta^3(x-y)\delta_{ab}$

$[\psi^\dagger, \psi^\dagger] = 0$

work

$[b_s(p), b^\dagger_{s'}(q)] = (2\pi)^3 2E_p \delta^3(\vec{p}-\vec{q})$

$[b, d^\dagger] = 0 = [b, d] = [b, b]$

$= [d, d]$

$(d, d^\dagger] = (2\pi)^3 2E_p \delta^3(\vec{p}-\vec{q})$

Insert $\psi^\dagger, \psi$ into $\underline{H}$

$$\vec{\partial} \rightarrow i\vec{p}$$

$$(i\gamma\cdot\partial + m)\, e^{-i\vec{p}\cdot\vec{x}}\, u(p)$$

$$= (\vec{\gamma}\cdot\vec{p} + m)\, e^{ip\cdot x}\, u(p)$$

$$\underbrace{-\gamma^0 p^0 + \vec{\gamma}\cdot\vec{p}}_{-p_\mu\gamma^\mu} + m + p^0\gamma^0$$

$$\bar{u} = u^\dagger \gamma^0$$

$$e^{-i\vec{p}\cdot\vec{x}}\left(-(p_\mu\gamma^\mu - m) + p^0\gamma^0\right)u(p) = e^{-i\vec{p}\cdot\vec{x}}\, p^0\gamma^0 u(p)$$

$$p^0 \underbrace{u^\dagger u}_{2m} \to 2mp^0\, b^\dagger b$$

$$H = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left[E_p\, b^\dagger b(s,p) - E_p\, d d^\dagger(s,p) \right]$$

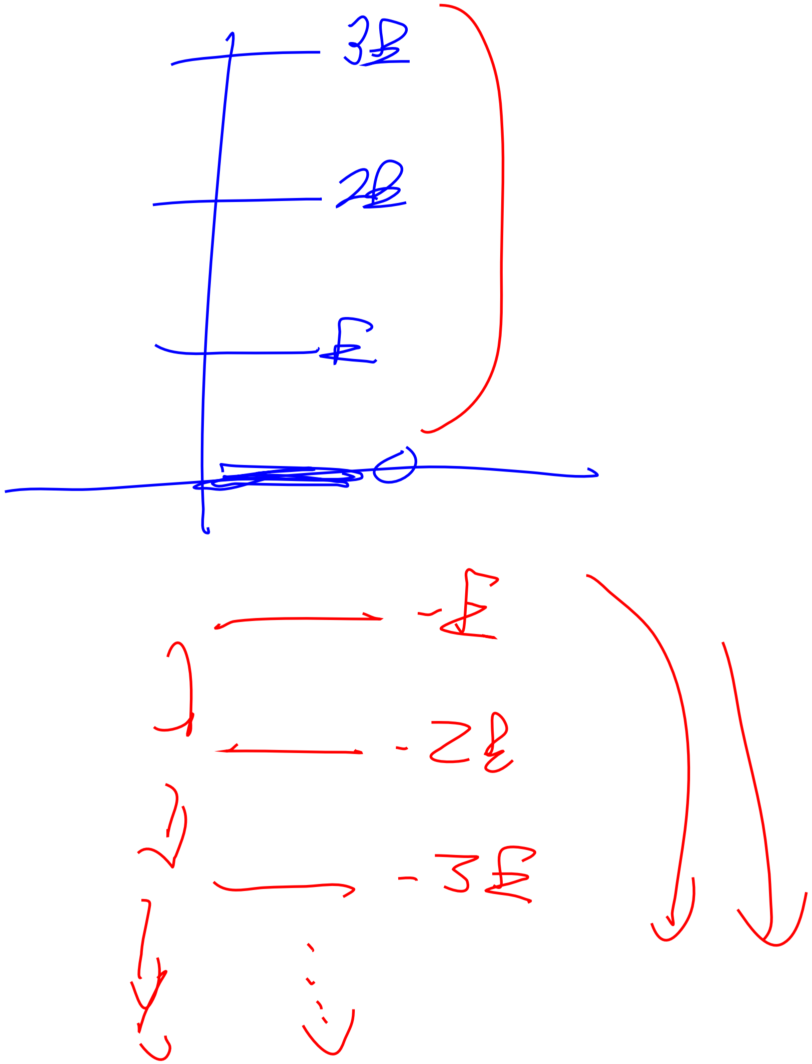
$(\not{p} - m)\, v = 0$ but for $p^0 < 0$

2 sets of SHO's.

$b^\dagger b \rightarrow$ positive energy

$\cancel{\dfrac{b^\dagger b + b b^\dagger}{2}}$ $\quad b^\dagger b |0\rangle = 0$

$dd^\dagger$ has EV's of $1, 2, 3, \ldots$



Assumed $[\psi^\dagger, \psi]$ unequal pts 0

$\{\psi, \psi\}$ " " "

Relativistic QFT $\to$ Spin Statistics Theorem

for spacelike $(x-y)$ $(x-y)_\mu (x-y)^\mu < 0$

for fields $\varphi(x)$ $\psi(y)$

Analytic
Unitary
4 Dim
Lorentz Inv.

$[\varphi(x), \psi(y)] = 0$ always if φ or ψ or both integer spin

$\boxed{\{\varphi(x), \psi(y)\} = 0}$ always if both φ, ψ odd-half int spin.

Proof Wightman & Streater pp 146-161