# Lecture 20: Fermions, Path Integral, Correlation Functions, and Scattering

Grassmann # (symbol)

$$Z(J(x), \bar{\eta}(x), \eta(x)) = \int \mathcal{D}\varphi \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \; \times$$

$\varphi \quad \psi \quad \bar{\psi}$

class. #     class ferm #'s

$\varphi$'s : $\varphi(x)$     + or − at each pt in sp.

$\varphi \quad \psi \quad \bar{\psi}$

$$\exp\left\{ i \int d^4x \left( \frac{\partial_\mu \varphi \partial^\mu \varphi}{2} - \frac{m_\varphi^2}{2}\varphi^2 - \frac{\lambda}{24}\varphi^4 + \bar{\psi}\boxed{(i\partial_\mu \gamma^\mu - m_\psi)}\psi - J(x)\varphi(x) - \bar{\eta}\psi - \bar{\psi}\eta \right)\right\}$$

$\langle \varphi \rangle$ then $\frac{i\delta}{\delta J(x)}$ puts $\varphi(x)$ in $\int$.

$\psi$ in path $\int$ : $i\frac{\delta}{\delta \bar{\eta}}$

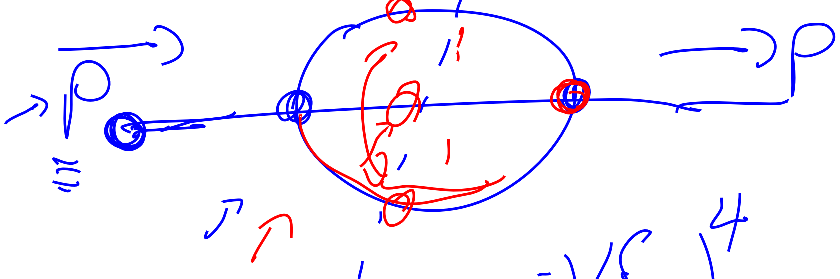
$\bar{\psi}$ in path $\int$ : $-i\frac{\delta}{\delta \eta}$

# Examples: Symmetry Factors

$\frac{\lambda\varphi^4}{24}$ thy $\quad 24 = 4!$

$\to 24$

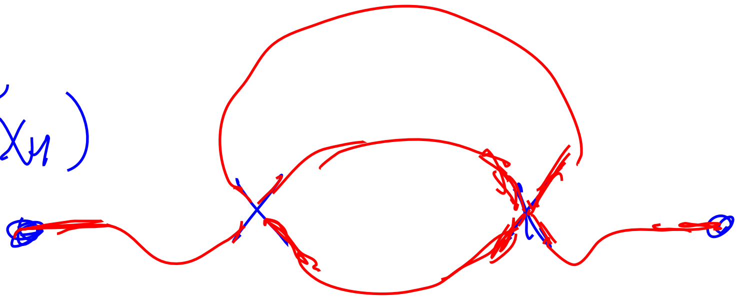
$\lambda^2$ corr. to 2-pt func.




$-i\lambda\left(\frac{\delta}{\delta J}\right)^4$

$e \qquad = \frac{1}{2} \times \times$

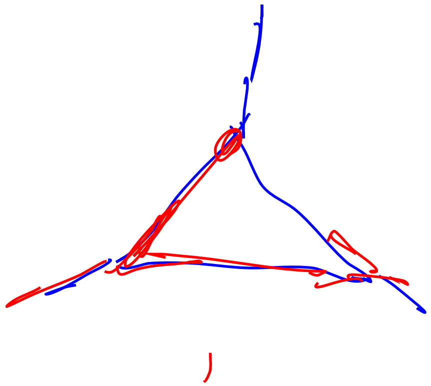
$\varphi(x_1)$ $\qquad\qquad$ $\varphi(x_2)$

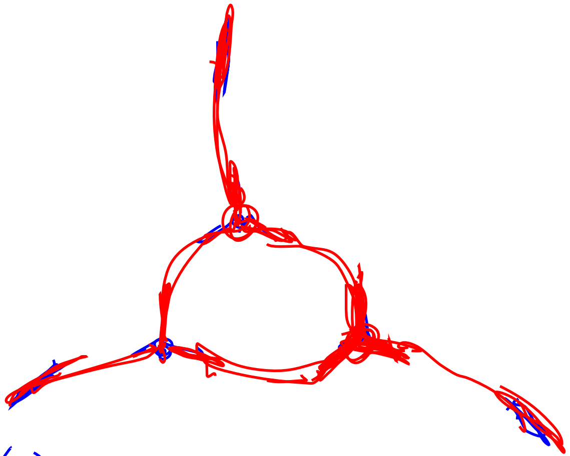


$$\frac{\lambda}{\cancel{24}_{\,3}}\;\frac{\lambda}{\cancel{24}}\;\frac{1}{2} \times \cancel{8}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1} = \boxed{\frac{\lambda^3}{6}}$$

sym 6

$\frac{\lambda \varphi^3}{6}$ th. y






$\frac{1}{\cancel{6}} \frac{1}{\cancel{6}} \frac{1}{\cancel{6}} \frac{1}{\cancel{6}} \cancel{4} \cdot \cancel{6} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{2} \cdot 1 = 1$

If I want $\langle T \varphi(x)\, \psi(y)\, \bar\psi(z) \rangle = \Theta(y^0 - z^0)\, \varphi\psi\bar\psi$

$\ominus\, \Theta(z_0 - y_0)\, \varphi\bar\psi\psi$

then compute

$$\frac{1}{Z}\frac{i\delta}{\delta J(x)}\frac{+i\delta}{\delta\bar\eta(y)}\frac{-i\delta}{\delta\eta(z)} Z(J,\eta,\bar\eta)$$

$J=0$

$\eta, \bar\eta = 0$

Relative

If I want $\langle T \varphi(x)\, \bar\psi(z)\, \psi(y) \rangle$ ??

$$\frac{1}{Z}\frac{i\delta}{\delta J(x)}\underbrace{\frac{-i\delta}{\delta\eta(z)}\frac{i\delta}{\delta\bar\eta(y)}}_{\text{anti-commute}} Z(J,\eta,\bar\eta)$$

$J, \eta, \bar\eta = 0$

$$\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\left(i\int_x \bar{\psi}(i\partial\!\!\!/ - m)\psi\right)$$

$$= \mathrm{Det}^{+1}\left[i(i\partial\!\!\!/ - m)\right] \qquad \text{scalar } \mathrm{Det}^{-1/2}$$

$1$ $\qquad$ $1/2$

Real scalars

Formally define $S_F(x-y)$ as $(i\gamma^\mu\partial_\mu - m)^{-1}(x-y)$

4x4 matrix $\qquad$ Matrix 4x4

huh? $\quad M^{-1}M = \mathbb{1}$

if $\quad F(x-y)\,G(y-z) = \delta^4(x-z)$ then $F = G^{-1}$

$$\frac{1}{4\pi^2 x^2} \to G(x-y)\left(\partial_\mu\partial^\mu - m^2\right) = \delta^4(x-y)$$

$$\to \frac{1}{p^2-m^2} \qquad p^2 - m^2 = 1$$

$$S_F(x-y)\left(i\overleftarrow{\partial}_\mu \gamma^\mu - m\right)_{y\to z} = \delta^4(x-z)\,\mathbb{1}_{4\times 4} \qquad \gamma^\mu p_\mu = \not{p}$$

p-space

$$(\not{p}-m)\,S(p) = \mathbb{1}$$

$$S(p) = \frac{1}{\not{p}-m} \quad \text{formal}$$

$$(\not{p}-m)\left[\frac{\not{p}+m}{\not{p}+m}\,\frac{1}{\not{p}-m}\right]$$

$$(\not{p}-m)^{-1}$$

$$(\not{p}+m)(\not{p}-m) = \not{p}\not{p} - \cancel{m\not{p}} + \cancel{m\not{p}} - m^2 = p^2 - m^2$$

$$\not{p}\not{p} = p^2$$

$$(\not{p}-m)\boxed{\frac{\not{p}+m}{p^2-m^2+i\epsilon}} = \mathbb{1}$$

$$S(p) = \frac{\not{p}+m}{p^2-m^2}$$

$S(x)$ = Fourier of

$$\underline{\langle 0| T(\varphi(x)\,\varphi(y))|0\rangle} = i\,G_F(x-y)$$

$$G_F(p) = \frac{1}{p^2 - m^2 + i\epsilon}$$

$$\langle 0| T(\psi(x)\,\bar{\psi}(y))|0\rangle = i\,S_F(x-y)$$

Free Thy

Prove it:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3\, 2E_p} \sum_s \left( b(p,s)\,u(p,s)\,e^{i\vec{p}\cdot\vec{x}} + d^\dagger v\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$e^{-ip_\mu x^\mu}$ $\qquad$ $e^{+ip_\mu x^\mu}$

$$\bar{\psi}(\ ) \quad \cdots$$

$$\boxed{\sum_s u\bar{u}(p,s) = \not{p} + m}$$

$$Z = \int \mathcal{D}\psi\, \mathcal{D}\bar\psi \; \exp\Big[ i \int d^4x \Big[ \cancel{g\varphi\bar\psi\psi} + \underbrace{\bar\psi(i\partial\!\!\!/ - m)\psi}_{S_F^{-1}} - \bar\eta\psi - \bar\psi\eta \Big] \Big]$$

$$\exp i \int g\, \frac{\delta}{\delta J}\, \frac{\delta}{\delta\eta}\, \frac{\delta}{\delta\bar\eta}$$

Scalars $\displaystyle \int \mathcal{D}\varphi \exp i \int \varphi\, \underbrace{(-\partial_\mu\partial^\mu - m^2)}_{G_F^{-1}}\, \frac{1}{2}\, \varphi - \cancel{\frac{\lambda}{24}\varphi^4} - J\varphi$

$$\frac{\delta}{\delta J} \cdots$$

$$\exp \frac{i}{2} \int_{xy} J\, G_F\, J$$

$G_F$: inv. of $-\partial_\mu\partial^\mu - m^2$

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$$\bar\psi(S_F^{-1})\psi - \bar\eta\psi - \bar\psi\eta = (\bar\psi - \bar\eta S_F)(S_F^{-1})(\psi - S_F\eta) - \bar\eta S_F \eta$$

Shift int var. $\psi \to \psi - S\eta$

$\bar\psi \to \bar\psi \quad \bar\eta S$

$$\bar\psi S^{-1}\psi - \bar\eta S\eta$$

If $[A,B]=0$, $e^{A+B} = e^A e^B$

$\bar\psi S^{-1}\psi$ and $\bar\eta S \eta$ commute

$\eta \bar\psi \psi = -\bar\psi \eta \psi = +\bar\psi\psi\eta$

$$e^{\bar\psi S^{-1}\psi - \bar\eta S \eta} = e^{\bar\psi S^{-1}\psi} e^{-\bar\eta S \eta}$$

$$\int \mathcal{D}\psi \mathcal{D}\bar\psi \, e^{i[\bar\psi S^{-1}\psi - \bar\eta\psi - \bar\psi\eta]} = \underbrace{\int \mathcal{D}\psi \mathcal{D}\bar\psi \, e^{i\bar\psi S^{-1}\psi}} e^{i\bar\eta S \eta}$$

$$= \operatorname{Det} S_F^{-1} \underbrace{\exp -i \int \bar\eta(x) S_F(x-y) \eta(y) \, d^4x \, d^4y}$$

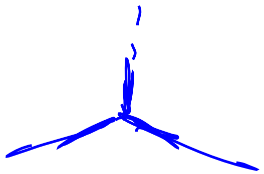
Scalars:

$$\frac{i}{p^2-m^2+i\epsilon} \quad \text{from} \quad e^{\frac{i}{2}J\frac{1}{p^2-m^2}J}$$

Spinors:

$$\frac{i}{\not p-m+i\epsilon} = \frac{i(\not p+m)}{p^2-m^2+i\epsilon} \quad \text{from} \quad e^{-i\bar\eta\frac{1}{\not p-m}\eta}$$

$$\lambda\, e^{-\frac{i\lambda}{24}\int d^4x\,\varphi^4} \to \left(\frac{\delta}{\delta J}\right)^4 =$$



$$e^{+ig\int d^4x\,\varphi\bar\psi\psi} \to e^{ig\int d^4x\left(\frac{i\delta}{\delta J}\right)\left(\frac{i\delta}{\delta\bar\eta}\right)\left(\frac{-i\delta}{\delta\eta}\right)}$$

Looks same

Simple Difference: $\psi, \bar{\psi}$ separate obj.

$$\frac{\delta}{\delta \bar{\eta}} \quad \frac{\delta}{\delta \eta} \quad e^{\bar{\eta} S_F \eta}$$

$$\boxed{\frac{\delta}{\delta \eta} \frac{\delta}{\delta \bar{\eta}}} e^{\bar{\eta} S_F \eta}$$

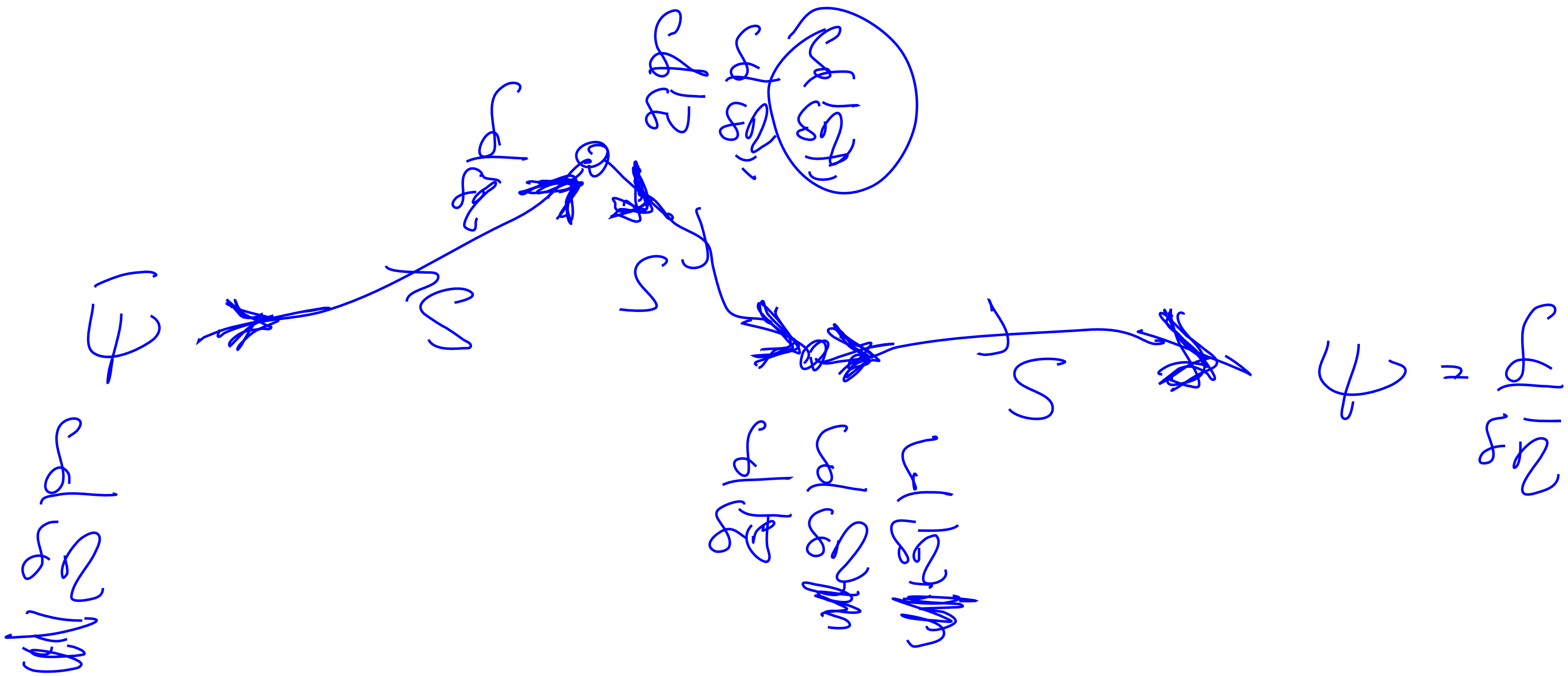
$$\frac{\delta}{\delta \eta} \left( -\bar{\eta} S_F \right) e^{\sim} = \left(-\bar{\eta} S\right)\left(-\bar{\eta} S\right) e^{\sim} \Big| = 0$$

$\frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}}$'s have to pair off

$\eta = 0$

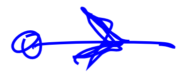
$\bar{\eta} = 0$

$S_F = \langle \psi \bar{\psi} \rangle$



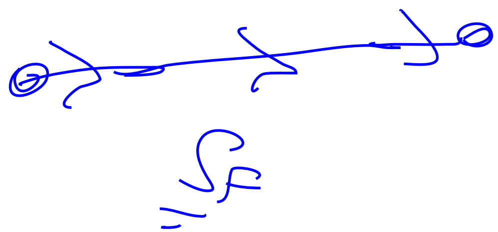

$\bar\psi \quad e^{\bar\eta S\eta}$

$$\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta}e^{\bar\eta S\eta} \qquad S\eta\, e^{\bar\eta S\eta}$$

$\frac{\delta}{\delta\eta}$ 

$\frac{\delta}{\delta\bar\eta}$ 

Propagators have arrows

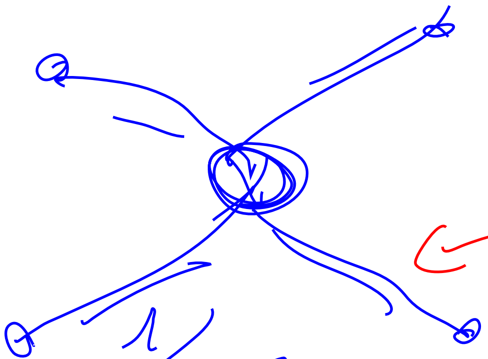


Scatter 2 spin-$\frac{1}{2}$ part $\to$ 2 spin-$\frac{1}{2}$ part.

$\mathcal{L}(\varphi, \bar{\psi}, \psi)$

$\langle 0 | T\, \psi_\alpha(-k_2)\, \psi_\beta(-k_1)\, \bar{\psi}_\beta(p_1)\, \bar{\psi}_\alpha(p_2) | 0 \rangle = (\ldots)\, \mathcal{M}$

Scalar $\langle 0 | T\, \varphi(-k_2)\, \varphi(-k_1)\, \varphi(p_2)\, \varphi(p_1) | 0 \rangle = (2\pi)^4 \delta^4(p_1+p_2-k_1-k_2) \, \dfrac{i\mathcal{M}}{(p_1^2-m^2)(p_2^2-m^2)(k_1^2-m^2)(k_2^2-m^2)}$



$\dfrac{1}{p^2-m^2}$

But now $\not{p}+m$ — what's this??

$p^2-m^2$

What if I had N scalars $\varphi_a$ $a = 1, \dots N$

$$\mathcal{L} = \frac{1}{2} \underline{M_{ab}} \, \partial_\mu \varphi_a \partial^\mu \varphi_b - \underline{V(\varphi)}$$

$$\langle 0 | T(\varphi_a(x) \, \varphi_b(y)) | 0 \rangle = i M^{-1}_{ab} G_F(x-y)$$

$$\int d^4(x-y) e^{ip\cdot(x-y)} \quad p\text{-space} = \frac{i M^{-1}_{ab}}{p^2 - m^2 + i\epsilon} \longleftarrow$$

$\varphi_a$ does not create properly-norm. 1 particle.

screwed-up-norm. 1-part.

spect func. $\underline{\rho_{ab}} = \underline{M^{-1}_{ab}} \, 2\pi \delta(p^2 - m^2)$

What to do? Solve EV problem for $M_{ab}$.

$N$ e-val's $\lambda_i$ $\quad i = 1 \ldots N \quad \lambda_i \in \mathbb{R}^+$

$\angle$ e-vect $\xi_{b,i}$ $\quad$ $i$: which e-vector, $b$: col. index

$$M_{ab}\,\xi_{b,i} = \lambda_i\,\xi_{a,i}$$

$$\hat{M}^{-1}_{ab}\,\xi_{b,i} = \lambda^{-1}\,\xi_{a,i}$$

$$\left[ M_{ab} \right] \left[ \xi_{b,i} \right] = \lambda_i \left[ \xi_{a,i} \right]$$

Define $\tilde{\varphi}_i = \sum_a \lambda_i^{1/2}\,\xi_{a,i}\,\varphi_a$

lin comb. of $\varphi_a$

$$\boxed{\langle 0 | T(\tilde{\varphi}_i\,\tilde{\varphi}_j) | 0 \rangle} = \cancel{\sum_a}\,\cancel{\sum_b}\,\underbrace{\lambda_i^{1/2}\,\lambda_j^{1/2}}\,\xi_{a,i}\,\xi_{b,j}\,\langle \varphi_a\,\varphi_b \rangle$$

$$\langle \varphi_a\,\varphi_b \rangle = i M^{-1}_{ab}\, G_F(x-y)$$

$$\boxed{= \delta_{ij}\; i G_F(x-y)}$$

$$\xi_{ai}\,\underbrace{M^{-1}_{ab}\,\xi_{b,j}}_{\lambda_j^{-1}\,\xi_{a,j}} = \underbrace{\xi_{a,i}\,\xi_{a,j}}_{\delta_{ij}}\,\lambda_j^{-1}$$

property norm'd orthog... scalars

Scatter 4 part's:

$$\langle 0 | T\left(\tilde{\varphi}_i(x)\,\tilde{\varphi}_j(y)\,\tilde{\varphi}_k(z)\,\tilde{\varphi}_l(w)\right)|0\rangle \xrightarrow{\delta^4(\Sigma p)} \frac{\underline{\underline{M}}}{\Pi(p^2-m^2)} \text{ directly}$$

$$= \sum_{abcd} \underbrace{\lambda_i^{1/2}\lambda_j^{1/2}\lambda_k^{1/2}\lambda_l^{1/2}\, \xi_{ai}\,\xi_{bj}\,\xi_{ck}\,\xi_{dl}}\ \underbrace{\langle 0 | T\left(\underline{\underline{\varphi_a\,\varphi_b\,\varphi_c\,\varphi_d}}\right)|0\rangle}$$

$$\underbrace{\frac{i M^{-1}_{aa'}}{p^2-m^2}\ \frac{i M^{-1}_{bb'}}{p_2^2-m^2}\ \frac{i M^{-1}_{cc'}}{k_1^2-m^2}\ \frac{i M^{-1}_{dd'}}{k_2^2-m^2}}\ (2\pi)^4\delta(\,)\left(\text{Amp - would-be } \underline{\underline{M\text{-element}}}\right)$$

$$\underbrace{\lambda_i^{1/2}\xi_{a_i i}\, M^{-1}_{aa'}} = \lambda_i^{1/2}\lambda_i^{-1}\xi_{a',i} = \lambda_i^{-1/2}\xi_{a',i}$$

$$M = (\underline{\underline{\lambda^{-1/2}\xi}})(\underline{\underline{\lambda^{-1/2}\xi}})(\underline{\underline{\lambda^{-1/2}\xi}})(\underline{\underline{\lambda^{-1/2}\xi}})\ \underbrace{(\text{Amputated } \varphi\text{-}}_{}\ \text{corr. func.})$$

Summarize: If I want ext. state $\tilde{\varphi}_i$ properly-norm 1-part

Apply $\underbrace{\lambda_i^{-1/2}\, \xi_{ai}}$ $\varphi_a$ where $\lambda_i, \xi_{a,i}$ E-val / E-vect of $M_{ab}$

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For us: $S_F^{ab} = \dfrac{(\not{p}+m)_{ab}}{p^2-m^2+i\epsilon}$ "scalar" $\dfrac{M_{ab}^{-1}}{p^2-m^2+i\epsilon}$

$\lambda \equiv$

$G_F$

$\lambda^{-1}, \xi$ E-val / E-vect of $\not{p}+m = \sum_s \underbrace{u(p,s)}\,\underbrace{\bar{u}(p,s)}$

$M_{ab}^{-1} = \sum_i \lambda_i^{-1}\, \xi_{a,i}\, \xi_{b,i}$

$= \sum_i \underbrace{(\lambda_i^{-1/2}\xi_{a,i})}\,\underbrace{(\lambda_i^{-1/2}\xi_{b,i})}$

$\lambda^{-1/2}\xi \longrightarrow u$ or $\bar{u}(p,s)$ which one?

or $v$ or $\bar{v}(p,s)$

Rule: Incoming $u(p,s)$ particle

$\rightarrow$ $v(p,s)$ antiparticle

Outgoing $\bar{u}(p,s)$ part

$\rightarrow$ $v(p,s)$ antipart.

$v$ - E vector for $p^0 < 0$ sol'n.

$p^0 > 0$ in part

$p^0 < 0$ out part

$\not{p} + m$

$=$

$\bar{u} u$

$\bar{\psi}$ creates part

$\bar{\psi} = \int \bar{u} b^\dagger + \bar{v} d$

$\psi$ annih. part

$\int u b + v d^\dagger$

$=$ anti-[illegible]

$$\langle \underset{\smile}{\varphi}\,\underline{\varphi}\,\underline{\varphi}\,\underline{\varphi} \rangle^{\prime} \qquad \bar{\mu}^{\prime}\mu^{\prime}$$

$$\bar{u}^{\prime\prime}_{\mu}{}^{\nu}_{\rho}$$

$$\langle \bar{\psi}\psi\,\bar{\psi}\psi \rangle$$

$$\bar{u}$$