Calculations with Fermions

Learn through Examples

$\Rightarrow u, \bar{u}, \gamma^\mu, \dfrac{\not{p}+m}{p^2-m^2}$, Matrix stuff

$\Rightarrow$ Signs

Fermions - order of operations matters

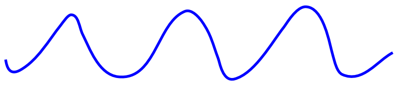
States have sign ambiguity.

$$|p_1 s_1 ; p_2 s_2\rangle = \underbrace{b^\dagger_{p_1 s_1} b^\dagger_{p_2 s_2}}_{A} |0\rangle$$

$$|p_2 s_2 ; p_1 s_1\rangle = \underbrace{b^\dagger_{p_2 s_2} b^\dagger_{p_1 s_1}}_{-A} |0\rangle$$

Which is right? $|P_1S_1;P_2S_2\rangle$ or $|P_2S_2;P_1S_1\rangle$?

Doesn't matter if consistent

→ $\langle P_1S_1 P_2S_2|$  $|i\rangle$ = Amplit.

A + B

$\langle P_2S_2;P_1S_1|$ Bad!

$\langle P_1S_1 P_2S_2|$

Relative – WRT

A+B ?

A – B ?

B – A ?

~~$(A+B)^2$ Diff't~~ wrong

Rate = $|\text{Amplit}|^2$ = $(A-B)^2$ or $(B-A)^2$ same

Learn: when are there "surprise" physically relevant

- signs? - Memorize. Never Think Again

Specific Example: $\underbrace{\psi, \bar{\psi}}_{\text{Dirac}}$, φ real-valued scalar.

Free Field Thy

3+ fields
Interactions

Yukawa Thy
$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$
$$+ i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi - y\varphi\bar{\psi}\psi$$

Perturb

$$\overset{\parallel}{i\bar{\psi}\not{\partial}\psi}$$

$\varphi\psi \rightarrow \varphi\psi$

$\psi\psi \rightarrow \psi\psi$ - signs

$\varphi\varphi \rightarrow \varphi\varphi$ - $y^4 \sim \lambda$ loops - signs & loops.

$$-J\varphi - \bar\eta\psi - \bar\psi\eta$$

$$\underline{\underline{i\mathcal{L}}} = \underbrace{i\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{im^3}{2}\varphi^2}_{+p^2} + \underbrace{i\bar\psi(i\overset{p}{\not\partial} - m)\psi}\, - iy\varphi\bar\psi\psi - \frac{i\lambda}{4!}\varphi^4$$

Propagator

$$\frac{1}{2}J\left(\frac{i}{p^2-m^2+i\epsilon}\right)J$$

$J \quad \bar J$

Ferm Propagator

$$\bar\eta\left(\frac{i(\not p+m)}{p^2-m^2+i\epsilon} = \frac{i}{\not p - m}\right)\eta$$

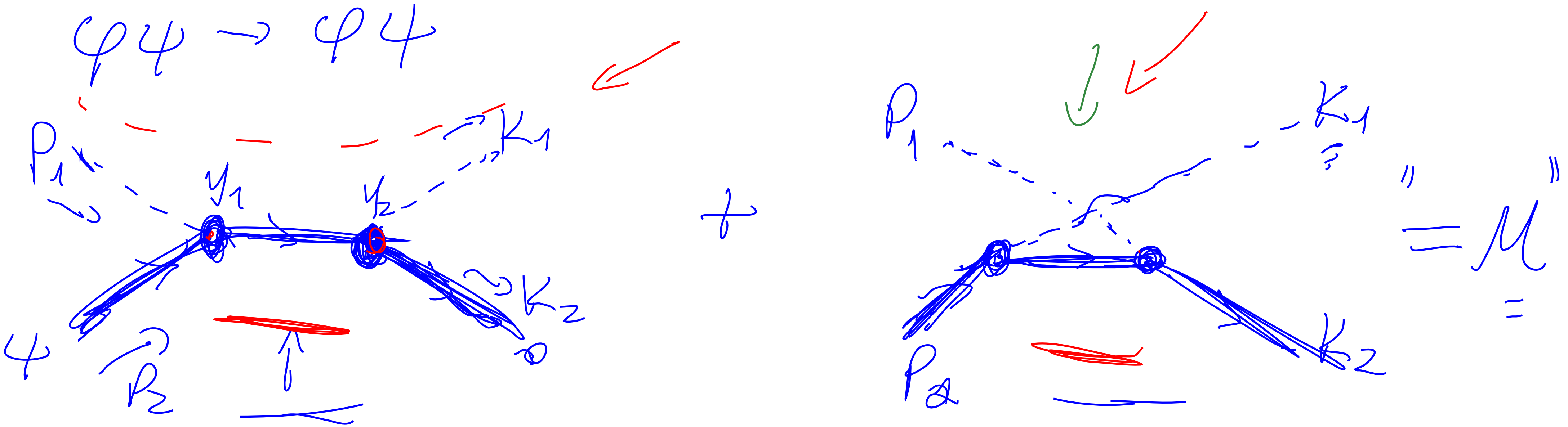
$$\eta \to \bar\eta$$

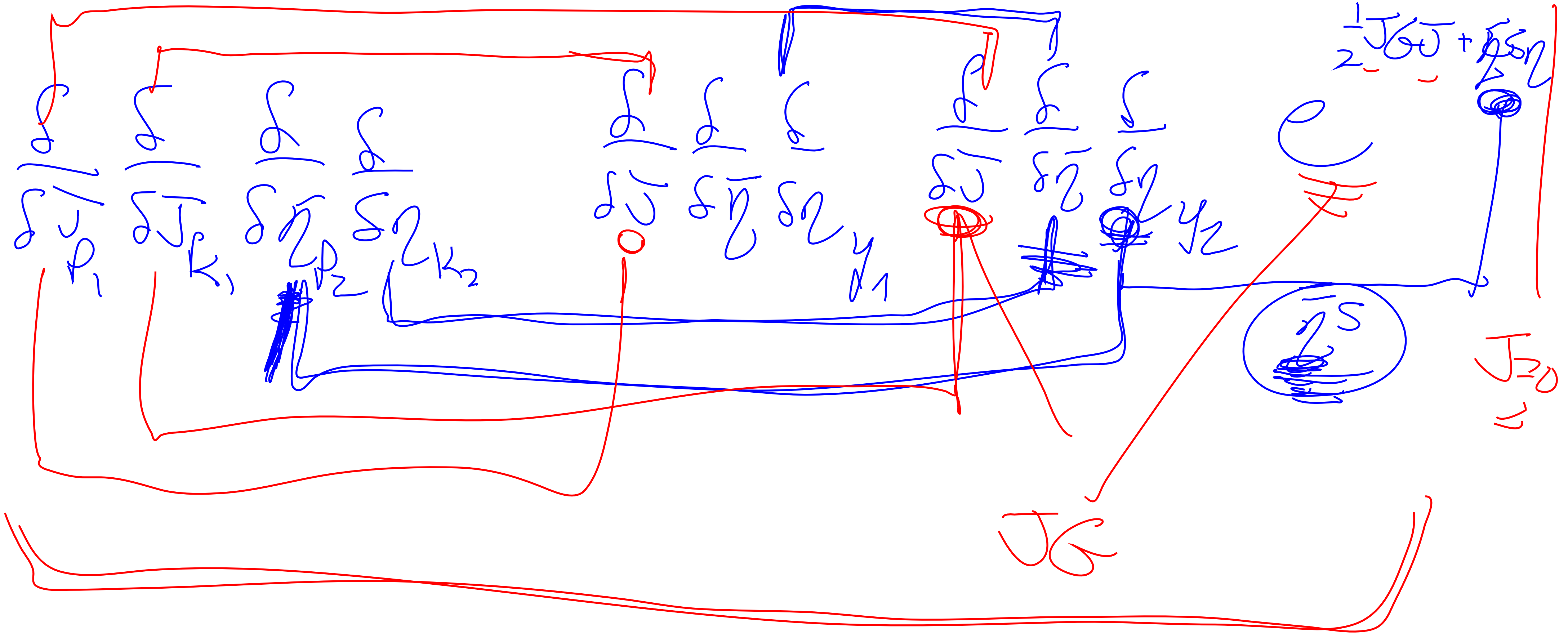
$$-iy$$

$$-i\lambda$$

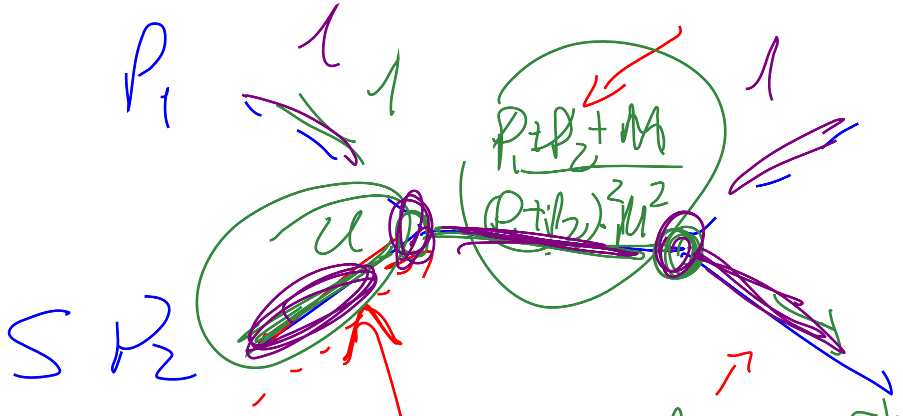
$\delta/\delta J$

$\frac{\delta}{\delta\eta} \quad \frac{\delta}{\delta\bar\eta}$





$$\frac{\delta}{\delta J_{p_1}}\frac{\delta}{\delta J_{k_1}}\frac{\delta}{\delta \bar\eta_{p_2}}\frac{\delta}{\delta \eta_{k_2}}\;\frac{\delta}{\delta J}\frac{\delta}{\delta \bar\eta}\frac{\delta}{\delta \eta}_{y_1}\;\frac{\delta}{\delta J}\frac{\delta}{\delta \bar\eta}\frac{\delta}{\delta \eta}_{y_2}\; e^{\frac{1}{2}JGJ+\bar\eta S\eta}$$



need as part of amp diag.

$\not p + m = u_{ps}\bar u_{ps}$

$\cancel{p^2-m^2+i\epsilon}$

→ Diagram

$$\langle 0|T(\overset{a}{\varphi}\overset{b}{\psi}\ \overset{b^\dagger a^\dagger}{\bar\varphi\bar\psi})|0\rangle$$

final initial

$$\bar\psi = \int_p \left(e^{-i\vec p\cdot\vec x} b^\dagger_s \bar u_{s,p} + e^{i\vec p\cdot\vec x} d\bar v\right)$$

creates part.

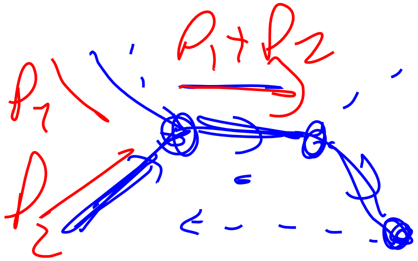
want: $\langle \text{finl}|$ Time-ev. $|\text{in}\rangle$

$\langle 0|a\,b$ = $\langle 0|\varphi\,\psi$

$a^\dagger b^\dagger|0\rangle$ = $\varphi\,\psi|0\rangle$

missing a $\bar u$ comp. to $\langle\varphi\psi\ \bar\varphi\bar\psi\rangle$

$$\boxed{i\mathcal{M} = \bar u(k_2,s')(-iy)\frac{\not p_1+\not p_2+M}{(p_1+p_2)^2-M^2+i\epsilon}(-iy)\,u(p_2,s)\cdot 1\cdot 1}$$



$$+i\mathcal{M} = +\bar{u}(k_2 s_2')(-iy)\frac{i(\not p_1+\not p_2+M)}{(p_1+p_2)^2-M^2+i\epsilon}(-iy)\,u(p_2 s) \quad \mathcal{M}_1$$

$$+\bar{u}(k_2 s')(-iy)\frac{i(\not p_2-\not k_1+M)}{(p_2-k_1)^2-M^2+i\epsilon}(-iy)\,u(p_2 s) \quad \mathcal{M}_2$$

Avg in, Sum final spin in computing $|\mathcal{M}|^2$??

$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ $\quad |\mathcal{M}|^2 = \mathcal{M}_1^*\mathcal{M}_1 + \mathcal{M}_2^*\mathcal{M}_2 + \mathcal{M}_1^\dagger\mathcal{M}_2 + \mathcal{M}_2^*\mathcal{M}_1$

$$\mathcal{M}_1 \mathcal{M}_1^* = -iy^2\, \bar{u}(k_2,s') \frac{\slashed{p}_1+\slashed{p}_2+m}{(p_1+p_2)^2-m^2}\, u(k,s)\, u^\dagger(k,s) \frac{\slashed{p}_1^\dagger+\slashed{p}_2^\dagger+m}{(p_1+p_2)^2-m^2}\, \bar{u}^\dagger(k_2 s')\, (+iy^2)$$

$$\bar{u} \quad \gamma^0 \frac{\slashed{p}^\dagger+\slashed{p}^\dagger+m}{(p_1+p_2)^2-m^2} \gamma^0 u$$

$$u^\dagger = \bar{u}\gamma^0$$

$$\gamma^0 \slashed{p}^\dagger \gamma^0 = \slashed{p}$$

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$$

$$\mathcal{M}_1 \mathcal{M}_1^* = y^4 \quad \mathrm{Tr}\, \frac{\slashed{p}_1+\slashed{p}_2+m}{(p_1+p_2)^2-m^2}\, u\bar{u}(k,s) \frac{\slashed{p}_1+\slashed{p}_2+m}{(p_1+p_2)^2-m^2}\, u(k_2,s')\bar{u}(k_2 s')$$

$$[\quad\quad][\quad\quad][\quad\quad][\quad\quad]$$

$$M_1^\dagger M_1 = y^4 \,\mathrm{Tr}\; \frac{\not p_1 + \not p_2 + M}{(p_1+p_2)^2 - M^2}\; \underline{\underline{u\bar{u}(k_2 s')}}\; \frac{\not p_1 + \not p_2 + M}{(p_1+p_2)^2 - M^2}\; \underline{\underline{u\bar{u}(\not p_2 s)}}$$

↑

Don't know initial state - Avg over poss. ←

$$\frac{1}{2} \sum_s$$

Don't know <u>final</u> state - Sum. You will see part, in either case.

$$1 \sum_{s'}$$

$$\frac{1}{2} \sum_s \underline{u\bar{u}(p_2, s)} = \underline{(\not p_2 + M)}\,\frac{1}{2}$$

$$1 \sum_{s'} u\bar{u}(k_2 s') = (\not k_2 + m)$$

$$\frac{1}{2}\sum_{ss'} M_1^\dagger M_1 = \text{Tr}\left[(\slashed{p}_1+\slashed{p}_2+m)(\slashed{k}_2+m)(\slashed{p}_1+\slashed{p}_2+m)(\slashed{p}_2+m)\right]\frac{1}{\left((p_1+p_2)^2-m^2\right)^2}$$

$$\left((p_1+p_2)^2-m^2\right)^2 = (s-m^2)^2$$

$$m^4 + m^3\,\cancel{(2\slashed{p}_1+3\slashed{p}_2+\slashed{k}_2)} + m^2(\slashed{p}\,\slashed{p}) \;\cancel{+\,m(\slashed{p}\,\slashed{p}\,\slashed{p})} + (\slashed{p}_1\slashed{p}_2\slashed{p}_3\slashed{p}_4)$$

$$m^4\,\text{Tr}\,\underset{4\times4}{1} = 4m^4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4$$

$\text{Tr}\,\slashed{p} = p_\mu \text{Tr}\,\gamma^\mu$ Each γ^μ is traceless. $\rightarrow 0$.

Tr (odd # of γ's) $= 0$.

$\text{Tr}\,ABC = \text{Tr}\,CAB$

$$\text{Tr}\,\gamma^\mu = \text{Tr}\,\underset{A}{\gamma^\mu}\underbrace{\gamma^5\gamma^5} = \text{Tr}\,\gamma^5\underset{A}{\gamma^\mu\gamma^5} = -\text{Tr}\,\gamma^\mu\gamma^5\gamma^5 = 0$$

$$\underbrace{\mathrm{Tr}\,\not{A}\not{B}} = A_\mu B_\nu \underbrace{\mathrm{Tr}\,\gamma^\mu\gamma^\nu}$$

$$\mathrm{Tr}\,\frac{1}{2}(\gamma^\mu\gamma^\nu + \underbrace{\gamma^\mu\gamma^\nu}_{\text{cyclic}}) = \mathrm{Tr}\,\frac{1}{2}\underbrace{(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)}_{g^{\mu\nu}\mathbb{1}} = g^{\mu\nu}\,\mathrm{Tr}\,\mathbb{1} = 4g^{\mu\nu}$$

$$= 4A_\mu B_\nu g^{\mu\nu} = \boxed{4A\cdot B}$$

$$\boxed{\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}}$$

$$\underbrace{\mathrm{Tr}\,\not{A}\not{B}\not{C}\not{D}} \text{ or } 2\,\mathrm{Tr}\,\underbrace{\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta} = \mathrm{Tr}\,\gamma^\mu\gamma^\nu\boxed{(\overset{2g^{\alpha\beta}}{\{\gamma^\alpha,\gamma^\beta\}} - \gamma^\beta\gamma^\alpha)}$$

$$= 2g^{\alpha\beta}\underbrace{\mathrm{Tr}\,\gamma^\mu\gamma^\nu}_{4g^{\mu\nu}} - \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\beta\gamma^\alpha$$

$$= 4(2g^{\alpha\beta}g^{\mu\nu} - 2g^{\mu\alpha}g^{\nu\beta} + 2g^{\nu\alpha}g^{\mu\beta}) - \underbrace{\mathrm{Tr}\,\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha}$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = 4(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\nu\alpha}g^{\mu\beta})$$

$$\mathrm{Tr}\, \not{A}\not{B}\not{C}\not{D} = 4\left(\underbrace{A\cdot B\, C\cdot D - A\cdot C\, B\cdot D + A\cdot D\, B\cdot C}_{3\ \text{terms}} \right)$$

$$\mathrm{Tr}\, \not{A}\not{B}\not{C}\not{D}\not{E}\not{F} = 4\left(\ldots\ 15\ \text{terms} \right)$$

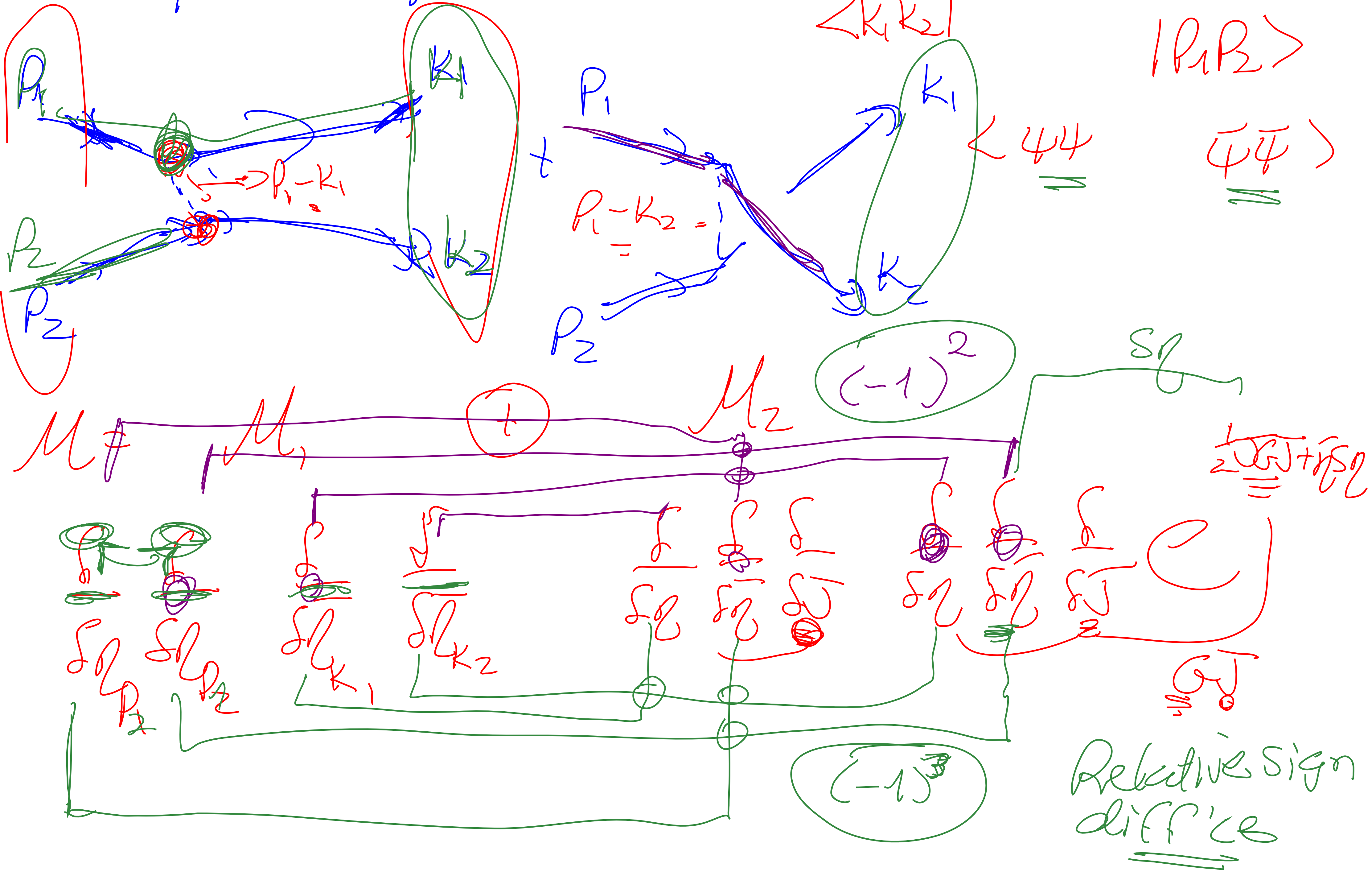
$$8 \qquad 7\cdot 5\cdot 3\ \text{terms}\ \ldots$$

$$\not{p}\not{p} = p^2 \mathbb{1}$$

$$\not{p}\not{A}\not{p} = \#\, p\cdot A\, \not{p}\ \#\, \not{A}\, p^2$$

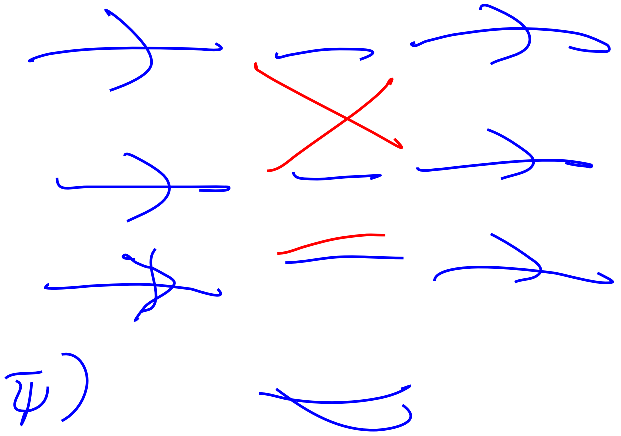
$$\vdots$$

44 → 44 Now what? 2nd order.



If ext. ferm. states connect together diff'tly

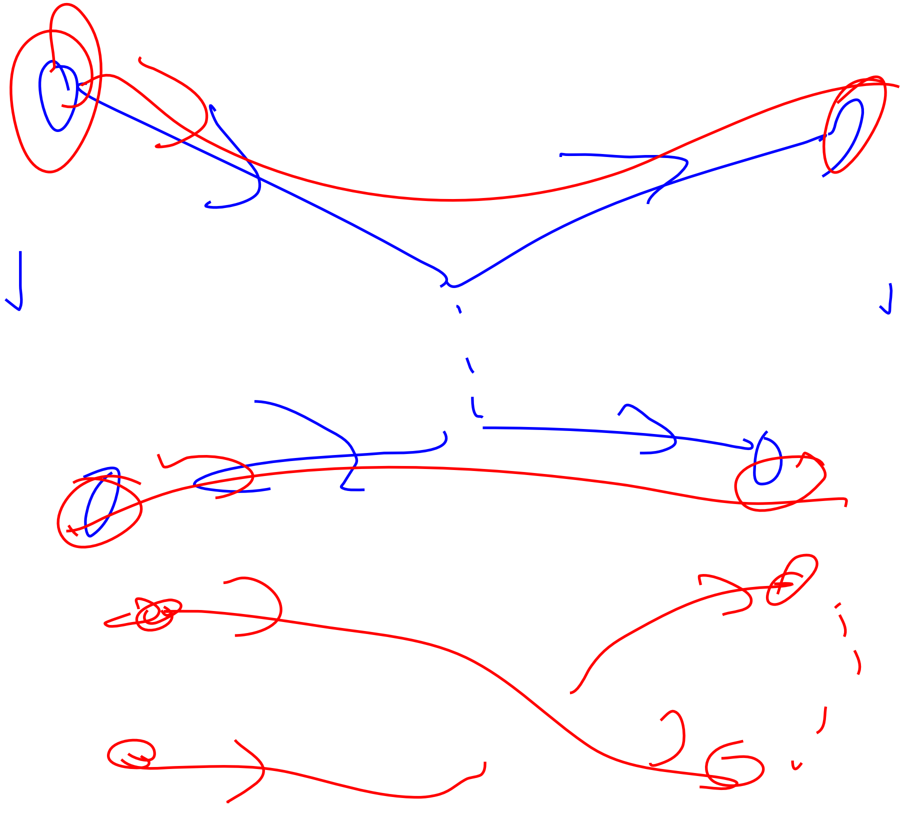
A in, spin ½ (in ψ, fin $\bar{\psi}$)

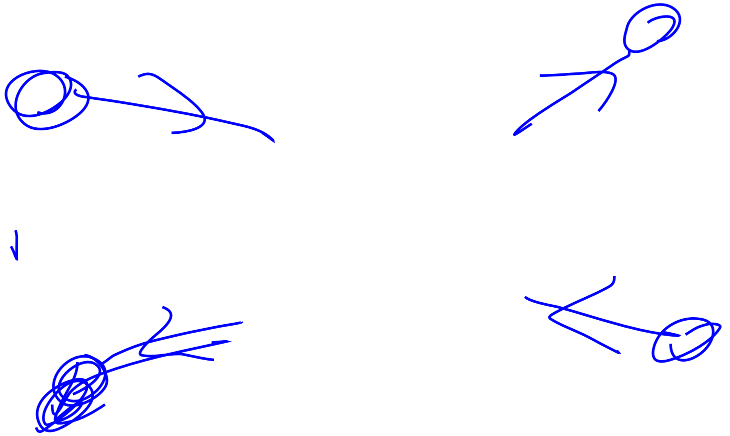


A outgoing (in $\bar{\psi}$ final ψ)

sign of permut. from one pairing to other controls (−) relative between M amp's.

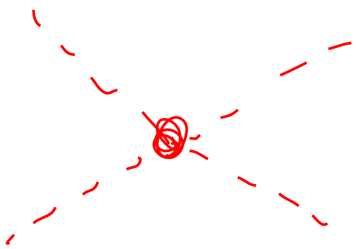
$\psi\bar{\psi} \to \psi\bar{\psi}$





What if $\varphi\varphi \longrightarrow \varphi\varphi$? But $y^4 \sim \lambda$

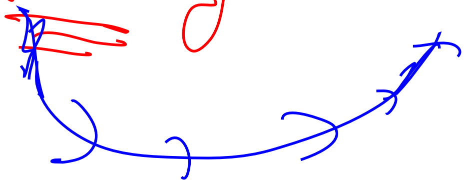
λ is tiny

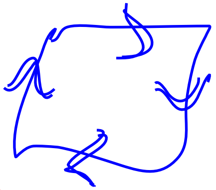


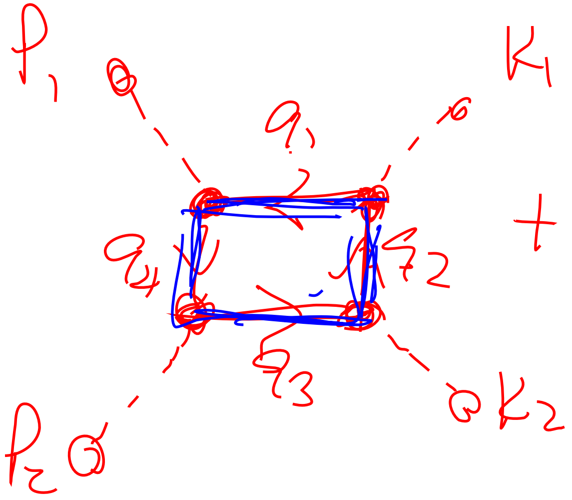
$\boxed{-i\lambda}$ tiny

$$\int \frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta\eta} e^{-\bar\eta S\eta} = S$$

$$\langle \psi \bar\psi \rangle = S$$







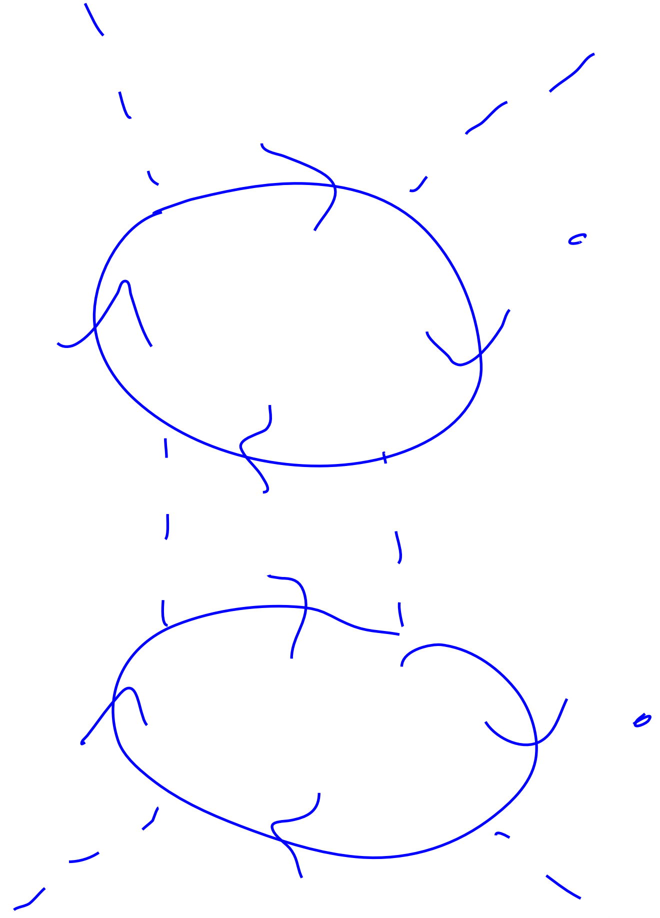
+ 5 permut.

$$\mathrm{Tr}\left(\frac{i(\not q_1+m)_{ab}}{(q_1^2-m^2)}(-iy)\frac{i(\not q_2+m)_{bc}}{\sim}(-iy) \times \frac{i(\not q_3+m)_{cd}}{(\sim}(-iy)\frac{i(\not q_4+m)_{de}}{\sim}(-iy)\right)$$

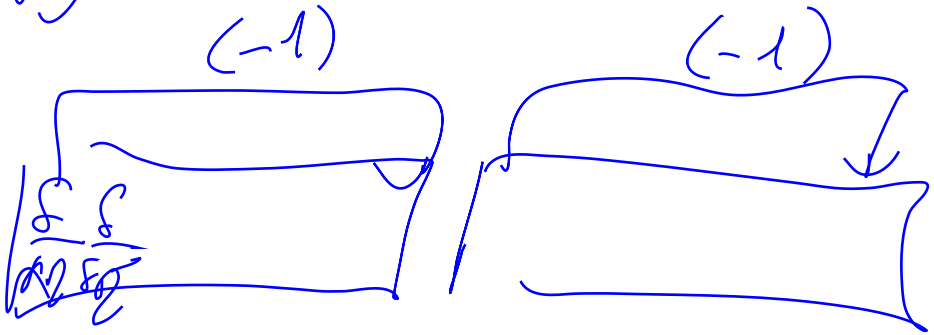
odd # - signs

$$\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta}\, e^{\bar\eta S\eta}$$

Always (−) sign for every fermionic loop



$(-1)^2$



Only - signs ever needed

$$\gamma^5 = \gamma_5 = \gamma_5 = \underline{i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$$\underline{\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5}$$

$$= \frac{i}{24}\epsilon_{\mu\nu\alpha\beta}\,\underline{\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta}$$

$$\mathrm{Tr}\,\gamma^5 = 0 \to \underline{\text{sum EV's} = 0} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}$$

J_i, K_i or $S_{\mu\nu}$

$$S_{\mu\nu} = \underline{\frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

$$\underline{(\gamma^5)^2 = \mathbb{1}} \quad \text{EV's are } \underline{\pm 1}\,. \quad 2\ \underline{+1}\text{'s} \quad 2\ \underline{-1}\text{'s}$$

$$\underline{\gamma^5 S_{\mu\nu} = S_{\mu\nu}\gamma^5}$$

$$\frac{1-\gamma^5}{2} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \to P_L \text{ keeps L kills R}$$

$$\frac{1+\gamma^5}{2} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \to P_R \text{ keeps R kills L}$$

$$L \begin{bmatrix} & 0 \\ 0 & \end{bmatrix} R$$

$$\underline{P_L^2 = P_L} \qquad \underline{P_R^2 = P_R} \qquad \boxed{P_L P_R = 0} \qquad \boxed{P_L + P_R = \mathbb{1}}$$