Back to Symmetries

We can now study theories with $\{\varphi, \bar\Psi \Psi\}$

But all interesting theories also have A_μ - 4-vector.

$A_\mu J^\mu$ — conserved current like EM current

In (classical E&M) Maxwell

$$\rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

symm

$$\epsilon_{\mu\nu\alpha\beta}\, \partial^\nu F^{\alpha\beta} = 0$$

anti

$$0 = \partial_\nu \partial_\mu F^{\nu\mu} = \boxed{\partial_\nu J^\nu}$$

$\nu \leftrightarrow \mu$

$$\cancel{\partial_\nu \partial_\mu \partial^\nu A^\mu} - \cancel{\partial_\nu \partial_\mu \partial^\mu A^\nu} = 0$$

To to have A_μ we need J^μ with $\partial_\mu J^\mu = 0$

Classically $\rightarrow$ Symm $\longrightarrow$ J^μ with $\boxed{\partial_\mu J^\mu = 0}$

Nöther Theorem

Full QFT level??

Step 1 — Understand Symm & Nöther Thm at Quantum level

Step 2 — Can I now build QFT w. A_μ and how?

$\longrightarrow$ QED, QCD, Standard Model

Explore Introduce

Example(s) of symmetry – Phase invariance

$$\Phi = \frac{\varphi_r + i\varphi_i}{\sqrt{2}} \quad \text{lin comb.} \qquad \Phi \in \mathbb{C}$$

$$\Phi^* = \frac{\varphi_r - i\varphi_i}{\sqrt{2}} \quad \text{lin comb.} \qquad \varphi_r, \varphi_i \text{ indep. real } \cancel{\text{fields}}$$

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \varphi_r \partial^\mu \varphi_r + \partial_\mu \varphi_i \partial^\mu \varphi_i\right]$$

$$= \boxed{\partial_\mu \Phi}\, \partial^\mu \Phi^* \quad (\text{is real})$$

$$+ \frac{m^2}{2}(\varphi_r^2 + \varphi_i^2) + \frac{\lambda}{8}(\varphi_r^2 + \varphi_i^2)^2$$

$$= m^2\, \Phi^* \Phi + \frac{\lambda}{2}(\Phi^* \Phi)^2$$

$$\frac{\delta \mathcal{L}}{\delta \varphi_r}, \ \frac{\delta \mathcal{L}}{\delta \varphi_i} \quad \text{or} \quad \frac{\delta \mathcal{L}}{\delta \Phi}, \ \frac{\delta \mathcal{L}}{\delta \Phi^*} \quad //$$

ψ and its cc. $\bar{\psi}$

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

$\dfrac{\delta \mathcal{L}}{\delta \bar{\psi}}$ and $\dfrac{\delta \mathcal{L}}{\delta \psi}$ are indep. things to do.

Each has a symmetry!

Symmetry?

infinitesimal

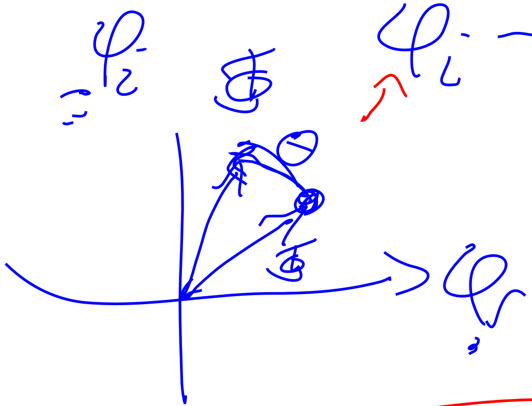
$\Phi \to e^{i\theta}\Phi$ or small θ,

$\Phi \to (1+i\theta)\Phi$

$\Phi = \frac{\varphi_r + i\varphi_i}{\sqrt{2}}$ $\quad \frac{\varphi_r + i\theta\varphi_r + i\varphi_i - \theta\varphi_i}{\sqrt{2}}$

$\varphi_r \to \varphi_r - \theta\varphi_i$

$\varphi_i \to \varphi_i + \theta\varphi_r$



$\Phi^* = (1-i\theta)\Phi^*$

$\mathcal{L}: \Phi\Phi^* \to \Phi \cancel{e^{+i\theta}} \Phi^* \cancel{e^{-i\theta}}$ unchg'd

$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$

$\psi \to e^{i\theta}\psi = (1+i\theta)\psi$

$\bar{\psi} \to e^{-i\theta}\bar{\psi}$

$\mathcal{L} \to \bar{\psi}\cancel{e^{-i\theta}}(i\partial\!\!\!/ - m)\cancel{e^{i\theta}}\psi$

unchanged. $U(1)$ symm.

More Generally φ_a

$\varphi_a \to \varphi_a + i\theta_A T^A_{ab}\varphi_b$

A – index over set of symmetries

$a\,b$ – indices over set of fields

θ_A for each indep. symmetry

Classically if $\varphi_a \to \varphi_a + i\Theta_A T^A_{ab}\varphi_b$ symm

$\mathcal{L}(\varphi_a \to \quad)$ unchanged $\to$

$$J^\mu_A = i T^A_{ab}\varphi_a \Pi^\mu_b \qquad \Pi^\mu_b = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi_b)} \qquad \partial_\mu J^\mu_A = 0$$

class.

For our examples?

$\varphi_a \to \Phi$ or Φ^* $\quad \Pi^\mu, \Pi^{*\mu}$ $\quad \Pi^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi} = \partial^\mu\Phi^*$ $\quad \Pi^{\mu *} = \partial^\mu\Phi$

$$\boxed{J^\mu} = i(\cancel{\Theta})\Phi\,\partial^\mu\Phi^* + i(-\cancel{\Theta})\Phi^*\partial^\mu\Phi = \boxed{i(\Phi\,\partial^\mu\Phi - \Phi\,\partial^\mu\Phi^*)}$$

$\varphi_a = \Phi$ $\qquad\qquad \varphi_a = \Phi^*$

$\psi \to e^{i\theta}\psi$ $\quad \Pi^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi} = i\bar\psi\gamma^\mu.$ $\quad \bar\Pi^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\bar\psi} = 0$

$$J^\mu = i\Theta i\bar\psi\gamma^\mu\psi = (-)\boxed{\bar\psi\gamma^\mu\psi = J^\mu}$$

But $\partial_\mu \hat{J}^\mu = 0$?? $\hat{J}^\mu = i(\hat{\Phi}^* \partial_\mu \hat{\Phi} - \hat{\Phi} \partial^\mu \hat{\Phi}^*)$

operator not exactly

or $\hat{\bar{\psi}} \gamma^\mu \hat{\psi}$

but ....

$$Z = \int \mathcal{D}\Phi^* \Phi \; \exp i \int d^4x \, \mathcal{L}(\Phi, \partial_\mu \Phi)$$

or $\mathcal{D}\psi\bar{\psi}$

change variables in $\varphi_a \to \varphi_a + \Theta_A(x)\, i T^A_{ab} \varphi_b$

hermitian

orthogonal

Assume transform does not change $\mathcal{D}\Phi^*\Phi$ etc.

$\psi\bar{\psi}$

If I rotate L, R comp's of ψ opposite,

$e^{i\Theta\gamma^5}\psi \to$ may not be unchg'd.

If I do δ ... chg variables inside, value $\int$ not chgd.

$$0 = \frac{\delta}{\delta \theta_A(x)} \int \mathcal{D}\phi^\dagger \mathcal{D}\phi \; \exp i \int d^4x \, \mathcal{L}[\phi, \partial_\mu \phi]$$

$$\phi_a \to \phi_a + i\theta_A(x) T^A_{ab} \phi_b$$

$$\mathcal{L} \to \mathcal{L} + \underbrace{\delta \mathcal{L}}_{\text{not } 0 \to \theta_A(x)}$$

$$\delta e^{i\int d^4x \mathcal{L}} = \left(\delta \, i \int d^4x \mathcal{L}\right) e^{i\int d^4x \mathcal{L}}$$

$$0 = \int \mathcal{D}\phi^\dagger \mathcal{D}\phi \; e^{i\int \mathcal{L}} \left[i \int d^4x \, \frac{\delta \mathcal{L}}{\delta \theta} \right]$$

$$\delta \int d^4x\, \mathcal{L}(\varphi, \partial_\mu \varphi) = \int d^4x\; i\, T_{ab}^{A}\, \Theta_A\, \varphi_b \left(\frac{\delta \mathcal{L}}{\delta \varphi_a} \right) - i\, T_{ab}^{A} \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi_a}\, \partial_\mu(\Theta_A \varphi_b)$$

$$\partial_\mu(\Theta_A \varphi_b) = \Theta_A\, \partial_\mu \varphi_b + \varphi_b\, \partial_\mu \Theta_A$$

$$\int d^4x\; i\, \Theta_A\, T_{ab}^{A} \cancel{\left[\varphi_b \frac{\delta \mathcal{L}}{\delta \varphi_a} - \partial_\mu \varphi_b \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi_a} \right]} - \underbrace{i\, \Pi_a^{\mu}\, T_{ab}^{A}\, \varphi_b}_{J^\mu}\, \partial_\mu \Theta$$

0 symm.

$$0 = \int \mathcal{D}\Phi\, \mathcal{D}\Phi^*\, e^{i\int \mathcal{L} \dots}\; J^\mu\, \partial_\mu \Theta \qquad \text{for any } \Theta(x)$$

$$\langle \partial_\mu J^\mu \rangle = 0$$

What about

$\langle \underline{\varphi\varphi} \rangle$, eg, $\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2} Z = \cdots$??

$$\langle T(\varphi_a(x)\,\varphi_b(y))\rangle = \int \mathcal{D}\varphi \; \varphi_a(x)\,\varphi_b(y)\, e^{i\int d^4x\,\mathcal{L}[\varphi,\partial_\mu\varphi]}$$

symm

change of var. $\varphi_c \to \varphi_c + i\,\underline{\Theta_A T^A_{cd}}\,\varphi_d$

$$0 = \int \mathcal{D}\varphi \Big[\varphi_b(y)\, i\Theta_A(x)\, T^A_{ac}\,\varphi_c(x) + \varphi_a(x)\, i\Theta_A(y)\, T^A_{bc}\,\varphi_c(y) + \varphi_a(x)\varphi_b(y)\int d^4z\; \partial_\mu\Theta_A(z)\, J^\mu_A\Big] e^{i\int \mathcal{L}\, d^4z'}$$

$$J^\mu \varphi \, \Theta(\ldots) + \varphi J^\mu \Theta(\ldots)$$

$$\langle 0 | T \Big(\int \partial_\mu J^\mu_A(z) \, \varphi_a(x) \varphi_b(y) \Theta(z) \, d^4 z$$

$$+ \, i T^A_{ac} \, \varphi_c(x) \varphi_b(y) \Theta(x)$$

$$+ \, i T^A_{bc} \, \varphi_c(y) \varphi_a(x) \Theta(y) \Big) | 0 \rangle = 0 \quad \text{For All } \Theta_A(z)$$

$$\langle \partial_\mu J^\mu(z) \, \varphi_a(x) \, \varphi_b(y) \rangle = -\delta^4(z-x) \, i T^A_{ac} \langle \varphi_c(x) \varphi_b(y) \rangle$$

$$- \delta^4(z-y) \, i T^A_{bc} \langle \varphi_c(y) \varphi_a(x) \rangle$$

contact term

$$\partial_\mu J^\mu \quad 0 \quad \varphi \quad \rightarrow 0$$

$$\partial_\mu J^\mu \, \varphi \qquad \varphi$$

$$\approx \varphi$$

Now the fun part

Symmetry: $\psi \to e^{i\theta}\psi$ nothing changes. $\mathcal{L} \to \mathcal{L}$

$J^\mu = \bar{\psi}\gamma^\mu\psi$

But what if $\psi \to e^{i\theta(x)}\psi$??

Huge symmetry - sep. rot. angle at each pt. in x.

This is not a symmetry!

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \to \bar{\psi}e^{-i\theta}(i\gamma^\mu\partial_\mu - m)e^{i\theta}\psi$$

$$= \bar{\psi}\cancel{e^{-i\theta}}(-m\cancel{e^{i\theta}} + i\,i\gamma^\mu(\partial_\mu\theta)\cancel{e^{i\theta}} + i\gamma^\mu\cancel{e^{i\theta}}\partial_\mu)\psi$$

$$= \underbrace{\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi}_{} - \underbrace{\partial_\mu\theta\,\bar{\psi}\gamma^\mu\psi}_{\partial_\mu\theta\, J^\mu} \neq \mathcal{L}$$

Dang!

Let's make it a symmetry:

$\mathcal{L} = \mathcal{L}_{old} + A_\mu J^\mu$ And $\psi \to e^{i\theta}\psi$

$\boxed{A_\mu \to A_\mu + \partial_\mu \theta}$

$\to \mathcal{L} - \cancel{\partial_\mu \theta J^\mu} + A_\mu J^\mu + \cancel{\partial_\mu \theta J^\mu}$ unchanged. Symmetry!

Alternatively: $A_\mu J^\mu = \bar{\psi}\gamma^\mu A_\mu \psi$

$\bar{\psi}\gamma^\mu i\partial_\mu \psi$

$\boxed{\begin{array}{c}\text{Covariant Deriv.}\\ \partial_\mu \to \partial_\mu - iA_\mu \equiv D_\mu\end{array}}$

$\bar{\psi}(i\gamma^\mu(\partial_\mu - iA_\mu) - m)\psi \to \bar{\psi}e^{-i\theta}(i\gamma^\mu(\partial_\mu - iA_\mu - i\partial_\mu\theta) - m)e^{i\theta}\psi$

$i\partial_\mu\theta$

$\to \bar{\psi}\cancel{e^{-i\theta}}\cancel{e^{i\theta}}(i\gamma^\mu(\partial_\mu - iA_\mu) - m)\psi$ unchg'd

What am I doing? Just a trick?

$$Z(J,\bar{\eta},\eta,J^{\mu}_{ext}) = \int \mathcal{D}[\Phi^*\Phi\,\psi\bar{\psi}\,A^{\mu}]\, e^{iS[\Phi^*,\Phi,\bar{\psi},\psi,A^{\mu}]}$$

$\varphi \quad \psi \quad \bar{\psi}$

$$S = D_\mu\Phi^* D^\mu\Phi - m^2\Phi^*\Phi - \frac{\lambda}{2}(\Phi^*\Phi)^2 + \bar{\psi}(i\cancel{D} - m)\psi$$

$$- J\varphi - \bar{\eta}\psi - \bar{\psi}\eta - A_\mu J^{\mu}_{ext}$$

$(\partial - iA)$

$(\partial_\mu + iA_\mu)\quad(\partial^\mu - iA^\mu)$

Φ^* trans. opp. Φ

opp. A^μ in it.

$+\mathcal{L}_A$ what's this?

$$\frac{\delta}{\delta A^\mu} \to J_\mu = 0$$

Dumb

QED

$\mathcal{L}_A$ must not chg when $A_\mu \to A_\mu + \partial_\mu \Theta(x)$

Most things change!

$\partial_\mu A^\mu \to \partial_\mu A^\mu + 2 A_\mu \partial^\mu \Theta + \cancel{\partial_\mu \Theta \partial^\mu \Theta}$

$\partial_\mu A_\nu \to \cancel{\partial_\mu \partial_\nu \Theta} + \partial_\mu A_\nu$

$- \partial_\nu A_\mu \to - \cancel{\partial_\nu \partial_\mu \Theta} - \partial_\nu A_\mu$

$\partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \partial_\nu A_\mu$ unchanged $= F_{\mu\nu}$

$\mathcal{L}_A = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$ obeys symm.

$A \to eA$. $F_{\mu\nu} \to e F_{\mu\nu}$

$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$D_\mu = \partial_\mu - ieA_\mu$

e-charge.

QCD: bigger symmetry. $\psi_{a,i}$ $a = 1,2,3$ or rgb

$$\psi_{a,i} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_3 \end{bmatrix}\begin{matrix} L \\ \\ R \end{matrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} \qquad \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} \to \begin{bmatrix} U \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

a, i: spinor

$U \to e^{iT^A_{ab}\theta_A}$

T^A_{ab} are Herm
Traceless 3×3

$$\lambda^{123} = \begin{bmatrix} \sigma_{123} & 0 \\ 0\;0 & 0 \end{bmatrix}, \quad \lambda^{4} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \lambda^{5} = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad \lambda^{6} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & +1 & 0 \end{bmatrix} \quad \lambda^{7} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$\lambda^8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} / \sqrt{3} = \lambda^A \text{ Gell-Mann matrices}$$

$$T^A = \frac{\lambda^A}{2}$$

$$\psi \longrightarrow \exp(i\theta_A T^A)\psi$$

$$\mathcal{L} = \bar\psi_a \left(\delta_{ab}\, i\partial\!\!\!/ + A\!\!\!/_A T^A_{ab} - m\delta_{ab}\right)\psi_b + \frac{F^A_{\mu\nu} F^{\mu\nu}_A}{4g^2}$$

symm $\psi_a \to \psi_a + i\theta_A T^A_{ab}\psi_b \to U\psi$

$\bar\psi_a \to \bar\psi_a - i\theta_A \bar\psi_b T^{*A}_{ab} \to \bar\psi U^+$

$$\mathcal{L} \to \bar\psi\left[1 - i\theta T^A\right]\left[\left(i\partial\!\!\!/ - m\right)\delta_{ab} + \left(A T^B + \delta A T\right)\right](1 + i\theta T)\psi$$

T's don't commute

$\partial\theta$

but also $-\theta T\, AT + AT\, \theta T$ don't cancel

conchg'diff $\delta A^A_\mu = \partial_\mu \theta_A + f_{ABC} A^B_\mu \theta_C$

$[T^A, T^B] = i f^{ABC}\, T^C$ $\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + f_{ABC} A^A_\mu A^C_\nu$