Learned : Field Content of (QED)

Lagrangian $\longrightarrow$ Feynman Rules, Pert. Thy.

Calculate $\frac{d\sigma}{d\Omega}$ or $|M|^2$ for LO processes

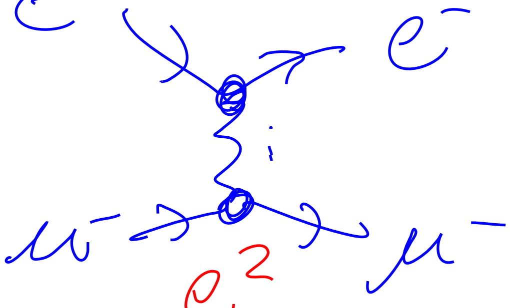
$e^-\mu^- \to e^-\mu^-$ crossings $e^-\gamma \to e^-\gamma$ $e^-e^+ \to \gamma\gamma$

$e^-e^- \to e^-e^-$

At Lowest Order in e^2

What about precise calc? e^2-suppressed effects?

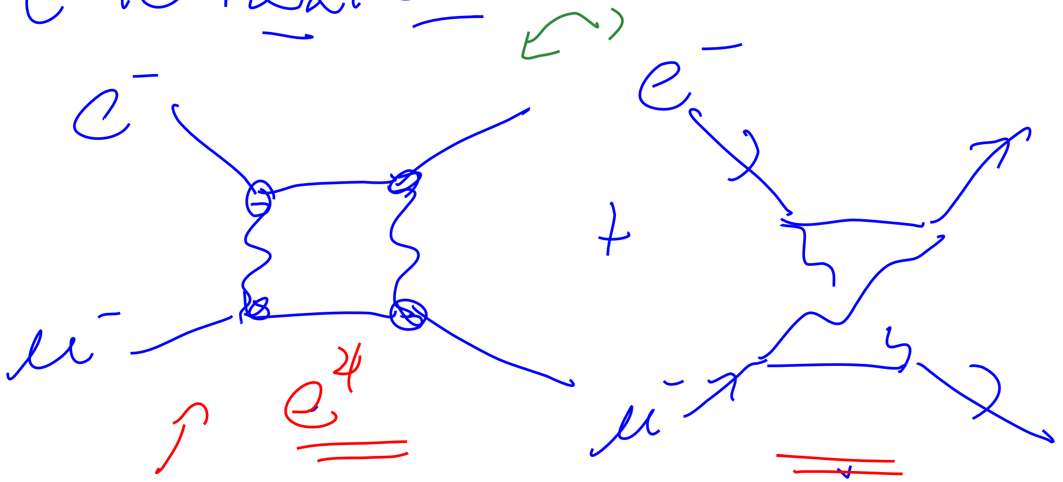
$e^-\mu^- \to e^-\mu^-$

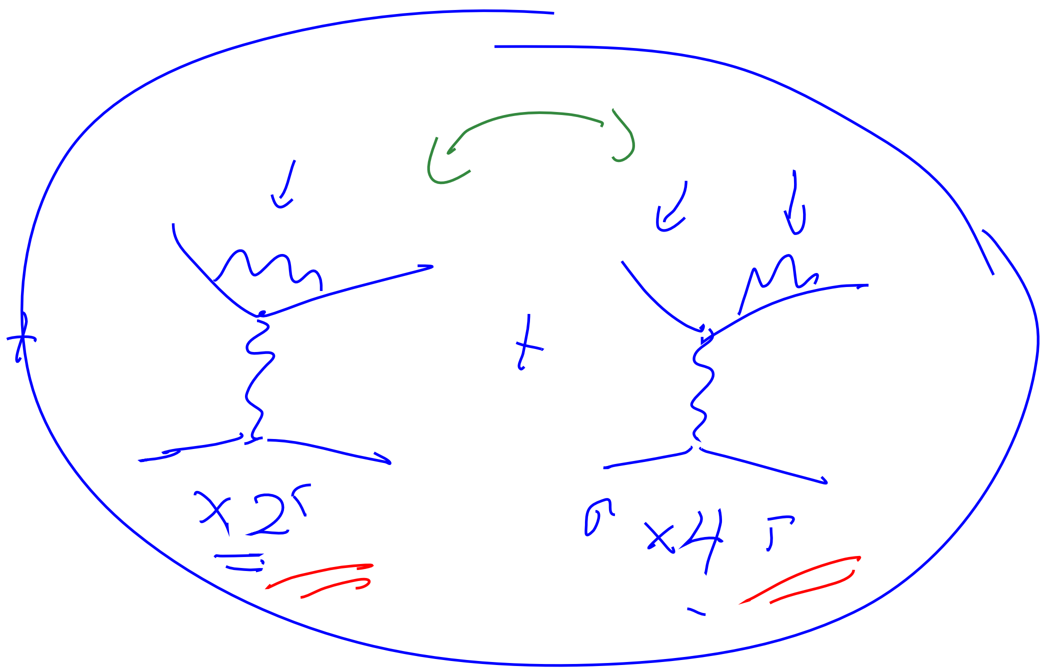


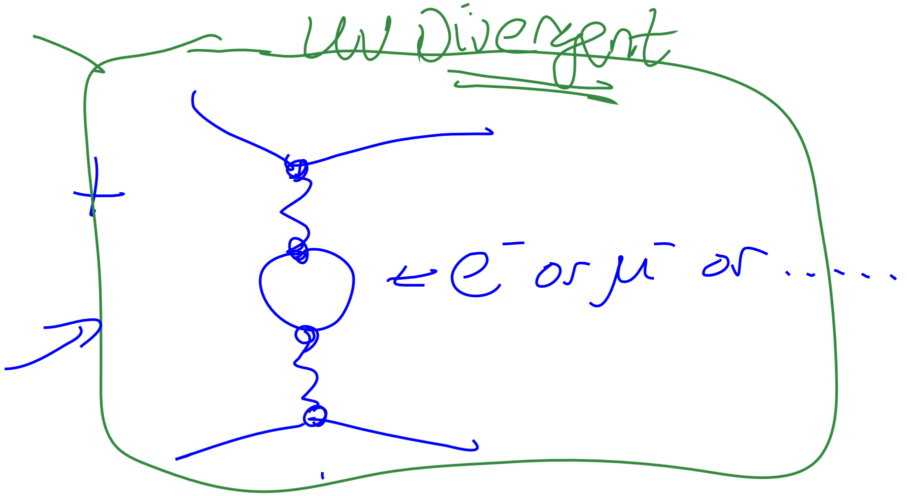
$\mathcal{M} \sim e^2$

$|\mathcal{M}|^2 \sim e^4$

$\mathcal{M}$ to next order







All cont. to $\mathcal{M}$ at $\mathcal{O}(e^4)$

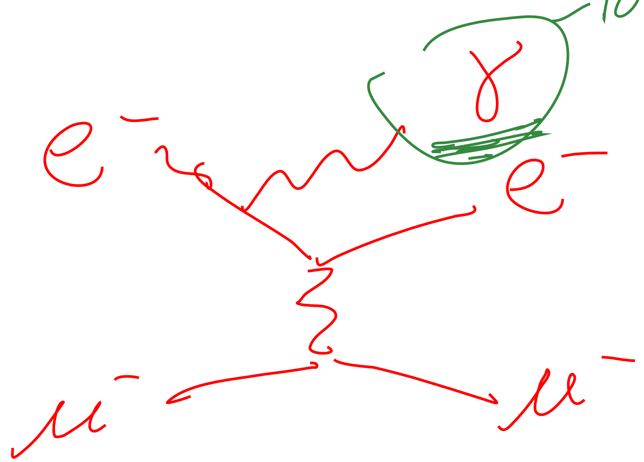
$\mathcal{M} = \mathcal{M}_2 + \mathcal{M}_4$

$$\frac{d\sigma}{d\Omega} = \int_{fin.p.} \dots \left| M_2 + M_4 \right|^2$$

interfer. e^3, e^4

$$\to e^4 M_2^* M_2 + e^6 \left(M_2^* M_4 + M_4^* M_2 \right) + e^8 \left(M_4^* M_4 + M_2^* M_6 + \dots \right)$$

Also e^3 processes?

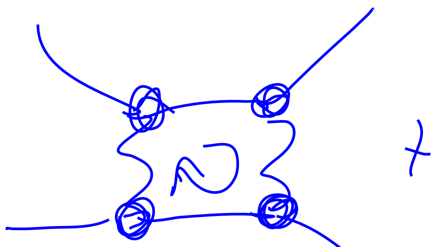


$e^- \mu^- \to \gamma e^- \mu^-$ Different process

$M \sim e^3 \qquad M^* M \sim e^6$

$\Downarrow$

$$\left. \frac{d\sigma}{d\Omega} \right|_{e^- \mu^- \to e^- \mu^- \gamma} \sim e^6 \dots$$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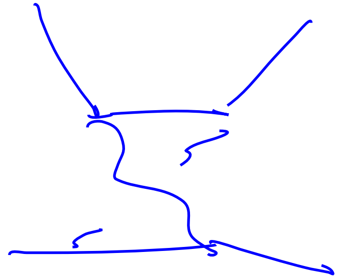
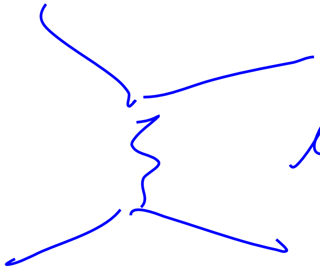


finite – $\int \frac{d^4 q}{(q^2+m^2)^3}$ UV finite IR

$\mathcal{M} \sim q_1^2 q_2^2 e^4$



$\mathcal{M} \sim q_1 q_2 e^2$ opp. sign for $e^-\mu^-$ as $\mu^+ e^-$

$|\mathcal{M}^+ \mathcal{M}| \sim q_1^2 q_2^2 e^4$ $\qquad \alpha^2$

$q_1 = -1$
$q_2 = -1$ $\quad e^-\mu^-$ Repuls.

$\mathcal{M}_2^* \mathcal{M}_4 \sim \underbrace{q_1^3 q_2^3}\, e^6 \sim \alpha^3$

cares about $q_1 = \pm q_2$

$q_1 = -1$
$q_2 = +1$ $\quad e^-\mu^+$ Attr.

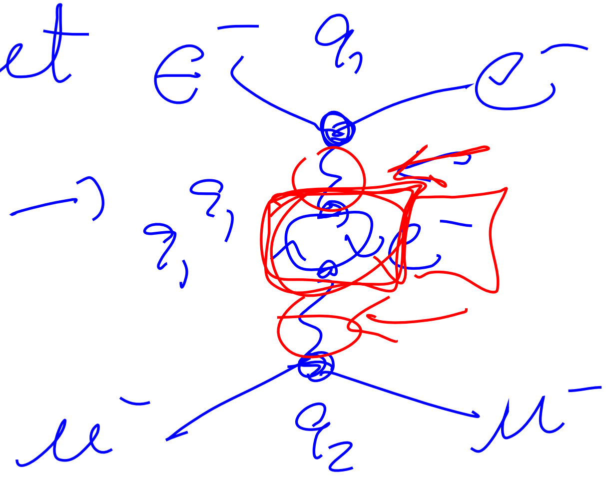
$$\frac{\mathcal{M}_2^* \mathcal{M}_4}{\mathcal{M}_2^* \mathcal{M}_2} \sim \frac{q_1 q_2 \alpha}{v}$$

Repulsive – destruct

Attract. – construct.

$v \sim \alpha \to \mathcal{O}(1)$

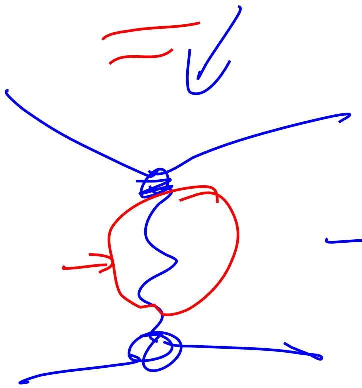
Talk about



$\sim q_1^3 q_2$ same sign as $q_1 q_2$

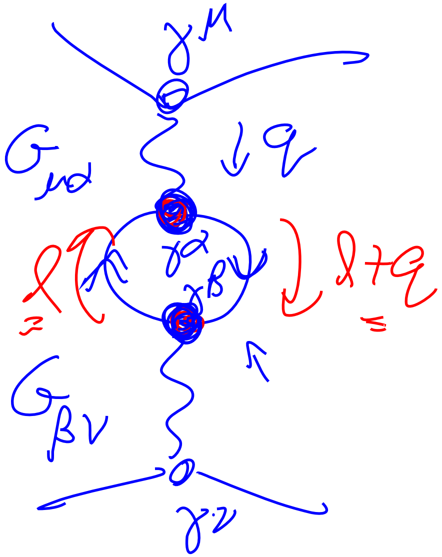
$$\begin{bmatrix} \bar{u}\gamma^\mu u \\ \bar{u}\gamma^\nu u \end{bmatrix} G_{\mu\alpha}\, i\Pi^{\alpha\beta} G_{\beta\nu}$$

A lot like



$$\longrightarrow \boxed{G_{\mu\nu}(q)} = \frac{-i\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)}{q^2 + i\epsilon}$$

$$\bar{u}\gamma^\mu u\; \bar{u}\gamma^\nu u\; G_{\mu\nu}$$



$$G_{\mu\nu} \to G_{\mu\alpha}(q)\, i\Pi^{\alpha\beta}(q)\, G_{\beta\nu}(q)$$

q_α

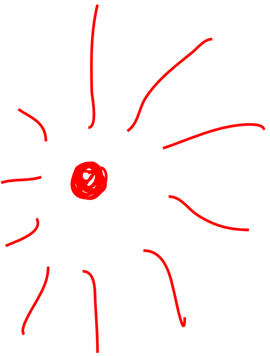
$$i\Pi^{\alpha\beta} = -(ie)^2 \int \frac{d^4 l}{(2\pi)^4}\, \mathrm{Tr}\left[\frac{i(\not l + m)}{l^2-m^2+i\epsilon}\,\gamma^\beta\, \frac{i(\not l+\not q+m)}{(l+q)^2-m^2+i\epsilon}\,\gamma^\alpha\right]$$

Self-energy

Claim: $q_\alpha \Pi^{\alpha\beta}(q) = 0$ "$\Pi^{\alpha\beta}$ is transverse"

$$\Pi^{\alpha\beta} \sim \langle J^\alpha J^\beta \rangle$$

Check: $\gamma^\alpha \to \not q$ Do I get 0? $q_\alpha \langle J^\alpha J^\beta \rangle \sim \langle \partial_\alpha J^\alpha J^\beta \rangle$



E^2+B^2

$$i q_\alpha \Pi^{\alpha\beta} = -(ie)^2 \int \frac{d^4 l}{(2\pi)^4}\, \mathrm{Tr}\left[i\,\frac{(\not l+\not q+m)}{(l+q)^2-m^2}\,(\not q+\not l-\not l-m+m)\,\frac{i(\not l+m)}{l^2-m^2}\,\gamma^\beta\right]$$

$(l+q)^2-m^2$ $\quad l^2-m^2$

$l \to l-q$

$$= -(ie)^2 \int \frac{d^4 l}{(2\pi)^4}\, \mathrm{Tr}\left[i\,\frac{i(\not l+m)}{l^2-m^2}\,\gamma^\beta - i\,\frac{i(\not l+\not q+m)}{(l+q)^2-m^2}\,\gamma^\beta\right]$$

We have

$$\underline{\underline{\int \frac{d^4 l}{(2\pi)^4}}} \, \mathrm{Tr}\left[\underline{\frac{\not{l}+m}{l^2-m^2}}_{\equiv} \gamma^\beta - \underline{\frac{\not{l}+\not{q}+m}{(l+q)^2-m^2}}_{\equiv} \gamma^\beta \right]$$

$$= \cancel{\int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr} \frac{\not{l}+m}{l^2-m^2} \gamma^\beta} - \underline{\int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr} \frac{\not{l}+\not{q}+m}{(l+q)^2-m^2} \gamma^\beta}$$

not here

here $l \to l-q$ $\quad dl \to \underline{\underline{d(l-q)}}$

$l+q \to l$

$$\cancel{\int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr} \frac{\not{l}+m}{l^2-m^2} \gamma^\beta}$$

$$= \underline{\underline{0}}$$

$d^4 l$ same as $\underline{d^4(l-q)}$ shift of $\int$ var's.

Gauge invariance Kinetic momentum physical

$$D_\mu = \partial_\mu - ieA_\mu \qquad e^{ik\cdot x}$$

$$i\left[k_\mu - eA_\mu\right]$$

Gauge change $\qquad \underbrace{\theta = x_\mu \xi^\mu}$ $\qquad$ ξ^μ const 4-vector

x_μ coord.

$$A_\mu \to A_\mu + \partial_\mu \theta = A_\mu + \xi_\mu$$

$k_\mu - eA_\mu$ const $\qquad k_\mu \to \underbrace{k_\mu + e\xi_\mu}$

Canon. Momentum

$\Pi^{\alpha\beta}(q)\, q_\alpha = 0$

$\Pi^{\alpha\beta}$ func. only of q.

Most general Rank-2 tensor dep. on q^μ only

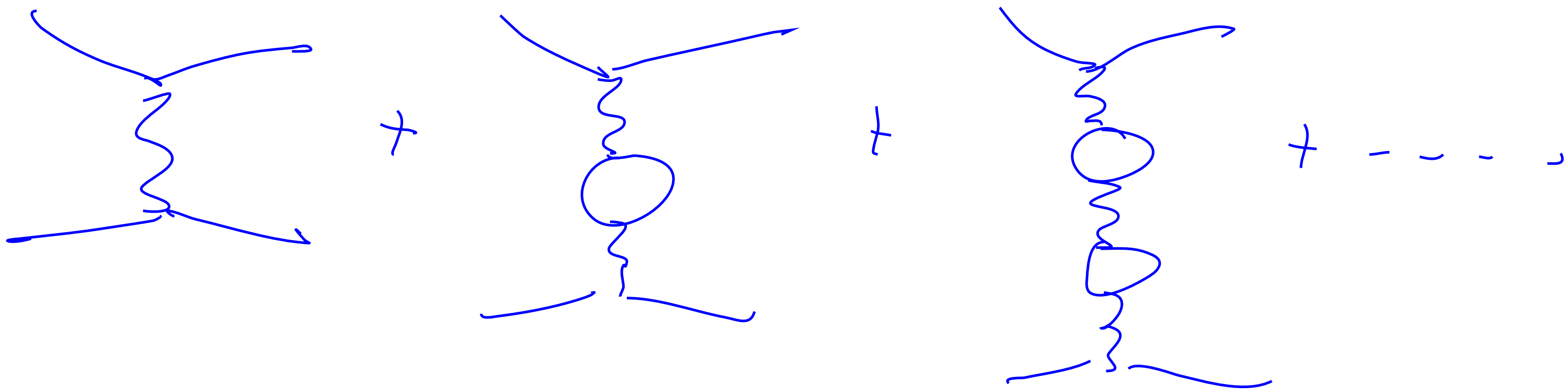
$$\Pi^{\alpha\beta}(q) = A(q^2)\, q^2 g^{\alpha\beta} + B(q^2)\, q^\alpha q^\beta$$

$$q_\alpha \Pi^{\alpha\beta} = A(q^2)\, q^2 q^\beta + B(q^2)\, q^2 q^\beta = 0 \qquad B = -A$$

$$\Pi^{\alpha\beta}(q) = \Pi(q^2)\left(q^2 g^{\alpha\beta} - q^\alpha q^\beta\right)$$

$\Pi(q^2 \to 0)$ nonsingular — no $1/q^2$ behavior

$$\lim_{q\to 0} \Pi^{\alpha\beta}(q) \to 0$$



$$\underbrace{G_{\mu\nu}(q) + G_{\mu\alpha}\, i\cancel{q^2}\Pi^{\alpha\beta} G_{\beta\nu} + G_{\mu\alpha}\, i q^2\Pi^{\alpha\beta} G_{\beta\gamma}\, i q^2\Pi^{\gamma\delta} G_{\delta\nu}} + \dots$$

$$-i\frac{\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)}{q^2} \qquad \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) = P_{\mu\nu}$$

$P_{\mu\nu}$ is proj. op: $P_{\mu\alpha} P^{\alpha}{}_{\nu} = P_{\mu\nu}$

$$P_{\mu\alpha} P^{\alpha}{}_{\nu} = \left(g_{\mu\alpha} - \frac{q_\mu q_\alpha}{q^2}\right)\left(g^{\alpha}{}_{\nu} - \frac{q^\alpha q_\nu}{q^2}\right)$$

$$P_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - \cancel{\frac{q_\mu q_\nu}{q^2}} + \cancel{\frac{q_\mu q_\nu}{q^2}}$$

$$\frac{-iP_{\mu\nu}}{p^2} + \frac{-iP_{\mu\alpha}\, i\Pi q^2 P_{\alpha\beta}\,(-i)P_{\beta\nu}}{p^2 \quad p^2} + \frac{(-i)P\,(i\Pi q^2)(-iP)(i\Pi q^2 P)(-iP)}{q^2 \quad q^2 \quad q^2} = -iP_{\mu\nu}\left(\frac{1}{q^2} + \frac{\Pi q^2}{q^2 q^2} + \frac{\Pi^2 q^4}{q^2 q^2 q^2}\right)$$

$$-iP_{\mu\nu} \times \left[\frac{1}{q^2} + \frac{q^2\Pi(q^2)}{q^2 q^2} + \frac{q^2\Pi(q^2)\, q^2\Pi(q^2)}{q^2 q^2 q^2} + \dots \right]$$

$$= \frac{-iP_{\mu\nu}}{\left(q^2 - q^2\Pi(q^2)\right)}$$

$\Pi(q^2)$ shifts denominator $\dfrac{1}{q^2}$

becomes $\dfrac{1}{q^2(1-\Pi(q^2))}$

$$q_\alpha\, i\,\Pi^{\alpha\beta}(q) = -(ie)^2 \int \frac{d^4\ell}{(2\pi)^4}\, \mathrm{Tr}\left[\frac{i(\not\ell+m)}{\ell^2-m^2+i\epsilon}\,\gamma^\alpha\, \frac{i(\not\ell+\not q+m)}{(\ell+q)^2-m^2+i\epsilon}\,\gamma^\beta\right]$$

Diverges ??

Large ℓ: $\;\; d^4\ell \qquad \frac{\not\ell}{\ell^2}\;\frac{\not\ell}{\ell^2} \qquad 1/\ell^2$

$\qquad\qquad \ell^4$

Options

$\cancel{\int_0^\Lambda d^4\ell}$ Conceptually right - wrong in detail

Fails to have shift symm - not Gauge Inv.

Pauli Villars - works, but horrible. Fails for QCD

Lattice - works [after Wick Rot - see below] Nasty

~~Heat~~ Kernel - Schwinger Proper Time

Dim. Regularization ——— what you should learn.

UV diverg. less bad in fewer ST dim's

$\int d^4 l \to \int d^3 l$ or $d^2 l$ or . . .   Dim D as contin. var.

Analytic behavior in D

4 Dim $\to$ $D = (4 - \underline{\underline{2\epsilon}})$ dim $\lim_{\epsilon \to 0}$ Physical Answers

Careful <u>consistent</u> - works.

$\mathrm{Tr}\, \underline{\underline{\mathbb{1}}} = 4$ $\underbrace{g^{\mu\nu} g_{\mu\nu}}$ $= \underline{D}$ <u>not</u> 4

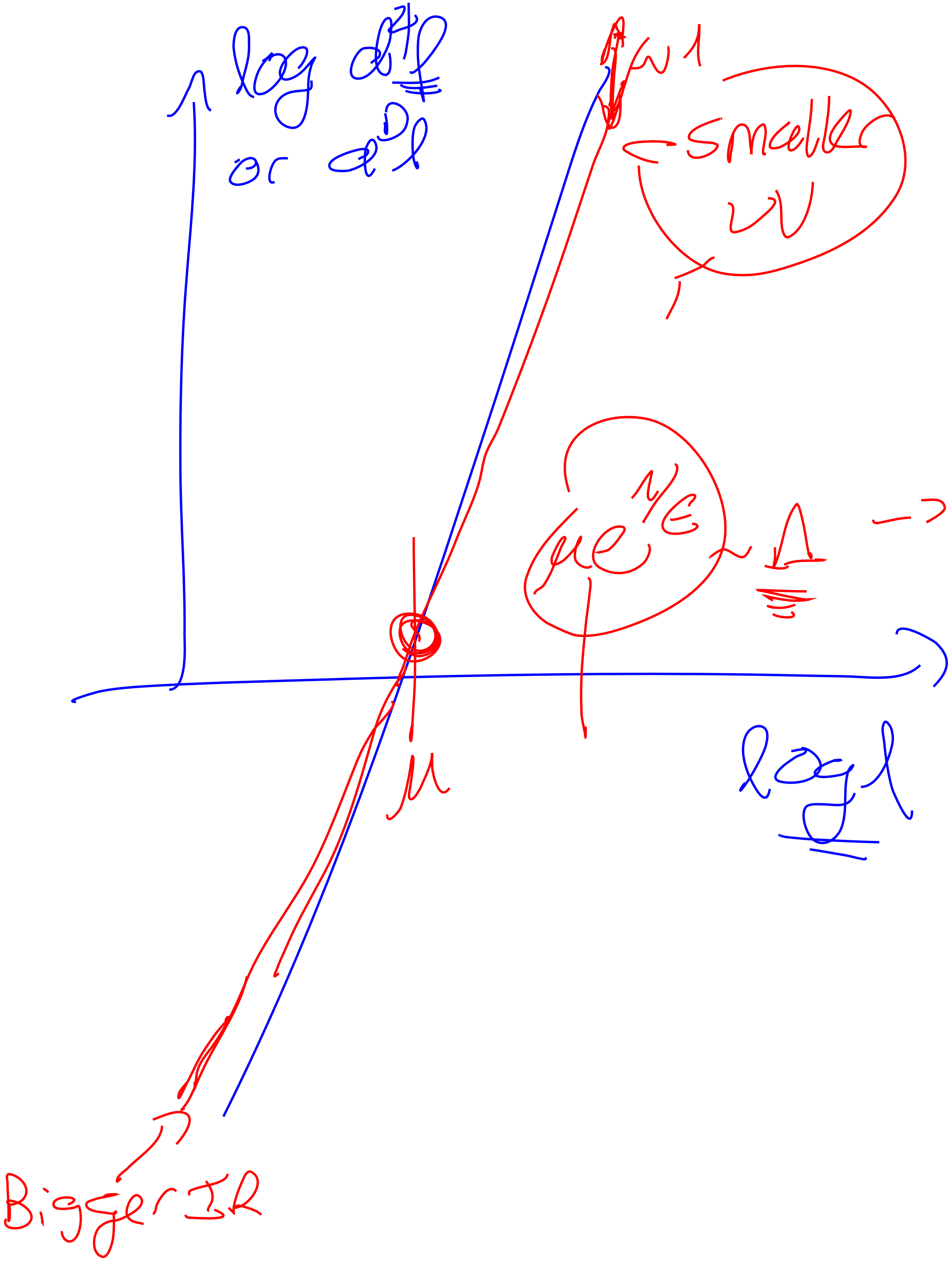
4x4 γ-matrix <u>land</u>

$\gamma^\mu \gamma_\mu = D$

$\gamma^\mu \not{l} \gamma_\mu = \underline{\underline{(2-D)}} \not{l}$

$\underbrace{\int \frac{d^4 l}{(2\pi)^4}} \to \underbrace{\mu^{4-D} \int \frac{d^D l}{(2\pi)^D}}$

<u>μ new scale</u> - to keep Dim of e^2 etc. <u>fixed</u>



$$\boxed{d^4\ell = d\Omega\, \ell^3 d\ell}$$

$$d^D\ell \qquad \ell^{D-1} d\ell$$

$$\to \log(\mu e^{1/\epsilon}) = \log\mu + \frac{1}{\epsilon}$$

$$\int \frac{d^4\ell}{\ell^4} \sim \log\Lambda^2 \sim \frac{1}{\epsilon}$$

$$\Pi^{\alpha\beta}(q) = \Pi(q)\left(q^2 g^{\alpha\beta} - q^\alpha q^\beta\right)$$

$$\gamma^\alpha \gamma_\alpha = D \text{ not } 4$$

$$\gamma^\alpha \gamma^\mu \gamma_\alpha = (2-D)\gamma^\mu$$

$$g_{\alpha\beta}\Pi^{\alpha\beta} = \Pi(q)\, q^2 [D-1]$$

$$i\,\Pi(q)\, q^2 (D-1) = (-1)(ie)^2 \int \frac{d^D l}{(2\pi)^D} \operatorname{Tr} \frac{\not l + m}{l^2 - m^2 + i\epsilon} \gamma^\alpha \frac{\not l + \not q + m}{(l+q)^2 - m^2 + i\epsilon} \gamma_\alpha$$

$$q^2 \Pi(q^2) = \frac{ie^2}{D-1} \int \frac{d^D l}{(2\pi)^D} 4 \left[\frac{Dm^2 + (2-D)\, l\cdot(l+q)}{(l^2 - m^2 + i\epsilon)((l+q)^2 - m^2 + i\epsilon)}\right]$$

$$" \frac{1}{A\cdot B} "$$

Feynman Trick

Proof: Do x-int.

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA+(1-x)B)^2}$$

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \frac{(n-1)!\,\delta(x_1+\cdots+x_n-1)}{(x_1A_1+\cdots+x_nA_n)^n}$$

$$\frac{1}{AB^2} = -\frac{d}{dB}\frac{1}{AB}$$

Integrand $\rightarrow \displaystyle\int dx \frac{Dm^2+(2-D)(\ell^2+\ell q)}{\big[\underbrace{x\ell^2-xm^2+(1-x)(q+\ell)^2-(1-x)m^2+i\epsilon}_{(\ell+(1-x)q)^2+x(1-x)q^2-m^2+i\epsilon}\big]^2}$

Shift $\ell \to \ell + (1-x)q$

$$q^2\pi = \frac{4}{D-1} i e^2 \mu^{4-D} \int_0^1 dx \int \frac{d^D \ell}{(2\pi)^D} \left[\frac{(2-D)\left(\ell^2 - x(1-x)q^2 + \cancel{(x-1/2)q\cdot\ell}\right) + Dm^2}{\left(\ell^2 - (m^2 - x(1-x)q^2) + i\epsilon\right)^2} \right]$$

even. $x \to 1-x$

odd

M^2 pos've.

even, $x \to 1-x$

$\ell^2 - M^2$

$\mp$ pos've

pos or neg

Do ℓ^0 int first.

$D = 1 + (D-1)$

time sp.

$g_{\mu\nu} = \begin{bmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & \ddots \end{bmatrix}$ $D-1$

$\ell^2 = \ell_0^2 - \vec{\ell}^2$

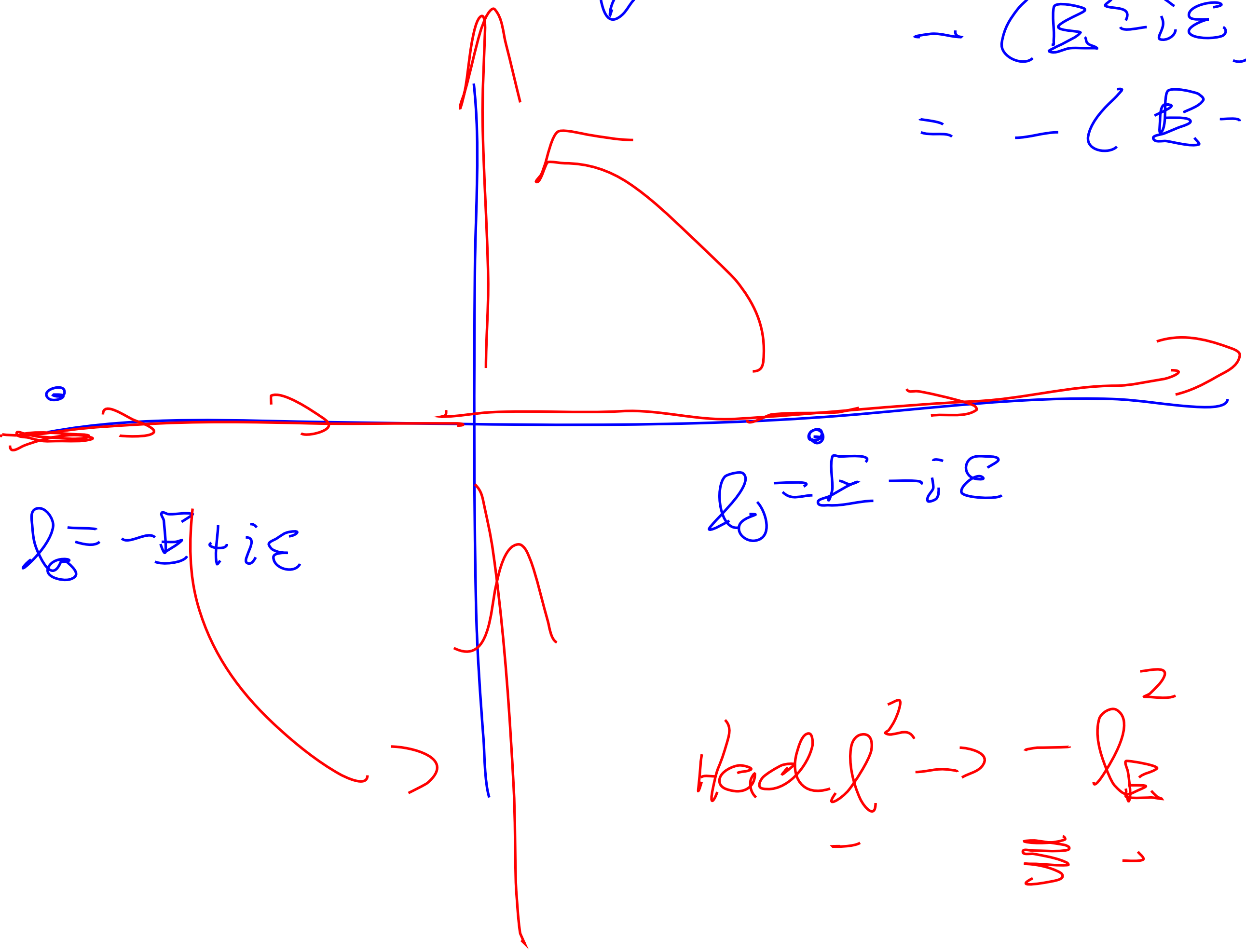
E^2

$\ell^2 - M^2 = \ell_0^2 - (\vec{\ell}^2 + M^2)$

l^0 int:

$$\int_{-\infty}^{\infty}\frac{dl^0}{2\pi}\,\frac{1}{\left(l_0^2-E^2+i\varepsilon\right)^2} \quad \text{or} \quad \frac{l_0^2}{\left(l_0^2-E^2+i\varepsilon\right)^2} \quad \text{or} \ldots$$

$$-\left(E^2-i\varepsilon\right) = -\left(E-i\varepsilon\right)^2$$



l^0 Rotates to il_E^0

$$= i\int\frac{dl_E^0}{2\pi}\,\frac{1}{\left(-l_E^{0\,2}-E^2\right)^2}$$

$$(l_E^0)^2+\vec{l}^{\,2}+m^2 = l_E^2$$

Had $l^2 \to -l_E^2$

Wick Rot.

$$q^2 \Pi = \frac{4 i e^2 \mu^{4-D}}{(D-1)} \int_0^1 dx \int \frac{d^D l_E}{(2\pi)^D} \frac{(2-D)\left(-\frac{l_E^2}{2} - x(1-x)q^2\right) + \ldots m^2}{(-l_E^2 + M^2)^2}$$

$$\frac{1}{2} \int (l_E^2)^{\frac{D-2}{2}} d(l_E^2)$$

$$\int \frac{d^D l_E}{(2\pi)^D} = \int l^{D-1} dl \int \frac{d\Omega_{D-1}}{(2\pi)^D}$$

Area of $D-1$-sphere

$$\int d^D x \, e^{-x^2/2} = \int x^{D-1} e^{-x^2/2} dx \int d\Omega_{D-1} = 2^{\frac{D-2}{2}} \Gamma\left(\frac{D}{2}\right) \int d\Omega_{D-1}$$

$$\left(\int dx \, e^{-x^2/2}\right)^D = \left(\sqrt{2\pi}\right)^D$$

$$\int d\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

Analytic Func

$$\int \frac{(\ell^2)^{\#} d(\ell^2)}{(\ell^2+m^2)^{\#'}} \longrightarrow \int_0^\infty (\ell^2)^n (\ell^2+m^2)^{-m} d(\ell^2) = (m^2)^{n+1-m} \int_0^1 dx\, x^{m-n-2}(1-x)^n$$

$$x = m^2/\ell^2+m^2$$

$$\rightarrow \int \frac{\ell^{2x-1} d\ell}{(\ell^2+m^2)^y} = \frac{(m^2)^{x-y}\Gamma(x)\Gamma(y-x)}{2\Gamma(y)} =$$

$$\rightarrow \Gamma(0)$$

$$(m^2)^{\frac{D-4}{2}} \frac{\Gamma(D/2)\Gamma(2-D/2)}{2\Gamma(2)} =$$

oldalags:

$$\rightarrow \int \frac{d^4\ell}{(\ell^2+m^2)^2} \quad D \rightarrow x = D/2 \quad y = 2$$

log Div.

$$(m)^{4-D}$$

$$\left(\frac{\mu}{m}\right)^{4-D}$$