Lecture 3 QFT I

HW I: 30 April
(Fridays)
HW meetings - Wed.

Quantum Field Thy - hard

Step: Classical Field Thy

Language: $S_{action} = \int d^4x \; \mathcal{L}[\varphi, \partial_\mu \varphi]$

spacetime

Lagrange density

Lorentz Invariance iff $\mathcal{L}$ is a scalar

(up to total deriv's)

$\mathcal{L} \to \mathcal{L}$ + total deriv. has no impact

More with scalar class Field Thy

Symmetries ⟶ exact predictions
(Nöther's Theorem)

Hamiltonian, canonical momenta, etc
[Extensive discussion of Lorentz group]

Ingredients needed in Quantum F. T.

Simpler classical

$\varphi(x)$ value at each pt in space

$\varphi : \sqrt{\text{Energy/Length}}$

$$\mathcal{L}[\varphi, \partial_\mu \varphi] = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{\omega_0^2}{2}\varphi^2 - \frac{\lambda}{24 \hbar}\varphi^4$$

$$g^{\mu\nu}\partial_\mu \varphi \partial_\nu \varphi$$

Lagr. Density, no Lagrangian

Euler-Lagrange $\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\right] = \frac{\delta \mathcal{L}}{\delta \varphi}$

∂_t and ∂_x!

$\boxed{\pi^\mu}$ Canonical 4-momentum density

$\pi^0 = \frac{\delta \mathcal{L}}{\delta \partial_0 \varphi}$ canon mom-density $\int d^3x\, \pi^0 = P$ canonical mom.

$$\partial_t^2 \varphi = -\omega_0^2 \varphi - \frac{\lambda}{6\hbar}\varphi^3 + \vec{\nabla}^2 \varphi$$

Hamiltonian?

$$L = \int d^3x\, \mathcal{L} = \int d^3x \left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)\right]$$

$$L = \int d^3x \left[\underbrace{\frac{1}{2}(\partial_t\varphi)^2}_{\text{Kinetic energy}} \underbrace{- \frac{1}{2}(\vec\nabla\varphi)^2 - V}_{\text{"potential" energy}}\right]$$

Energy from time change

Gradient Energy

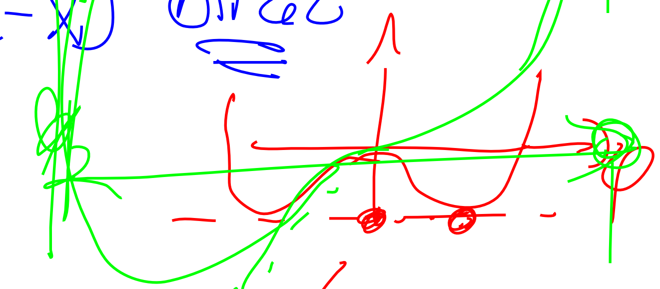
Energy of nonuniformity

Pot. Energy

En. due to nonzero value

$$\pi(x) = \frac{\delta L}{\delta\dot\varphi(x)} = \dot\varphi(x)$$

$$\frac{\delta\varphi(y)}{\delta\varphi(x)} = \delta^4(y-x) \quad \text{Dirac}$$



$$H = \sum P\dot Q - L = \int d^3x\, \pi(x)\dot\varphi(x) - L$$

$$= \int d^3x \left[\frac{1}{2}\pi^2 - \cancel{\frac{1}{2}\dot\varphi^2} + \frac{1}{2}(\vec\nabla\varphi)^2 + V(\varphi)\right]$$

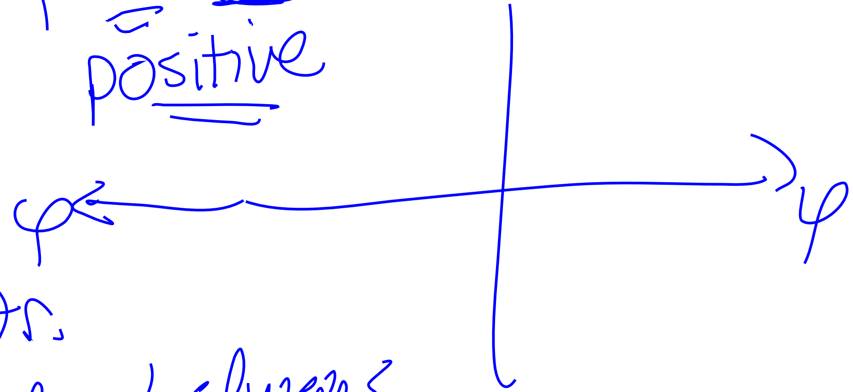
positive Kin. Pos. Grad. ~~Pos.~~ Pot. Bounded from below

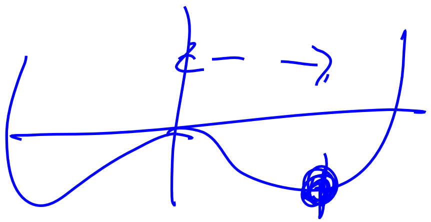
$$U(\varphi) = c_0 + \cancel{c_1\varphi} + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4 \ldots$$

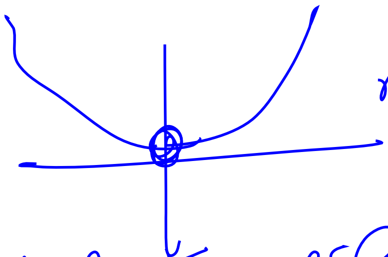
Positive if $\lambda = 0$ otherwise, either sign.

0 if $\lambda = 0$ otherwise, unconstr. Often 0. But not always

positive



neg:  weird

pos.  normal

Equivalent EOM.

Should I use $\mathcal{L}$ or H?

$$H = \int d^3x \, \mathcal{H}(\pi, \vec{\nabla}\varphi, \varphi)$$

$\mathcal{H}$ is not Lorentz-inv $+\frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\varphi|^2$

$\mathcal{H} = T^{00}$ comp. of Stress-Energy Tensor

$\mathcal{L}$ is scalar $\longrightarrow$ better, easier, etc. to formulate in terms of $\mathcal{L}$

$$\partial_t^2 \varphi = \underbrace{\nabla^2 \varphi - \omega_0^2 \varphi}_{\text{linear}} - \cancel{\frac{\bar{\lambda}}{6\hbar} \varphi^3} \quad \text{if you're lucky, } \lambda = 0$$

Use superposition $\varphi(x,t) = \varphi_1(x,t) + \varphi_2(x,t) + \ldots \ldots$

Postulate $\varphi(x,t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \tilde{\varphi}(\vec{p},t)$

$$\tilde{\varphi}(\vec{p},t) = \int d^3 x \, e^{-i\vec{p}\cdot\vec{x}} \varphi(x,t)$$

$$\int \frac{d^3 p}{(2\pi)^3} e^{ip\cdot x} \partial_t^2 \tilde{\varphi}(p,t) = \int \frac{d^3 p}{(2\pi)^3} (\underbrace{\nabla^2}_{-p^2} - \omega_0^2) e^{i\vec{p}\cdot\vec{x}} \tilde{\varphi}(p,t)$$

$$\partial_t^2 \tilde{\varphi}(p,t) = (-p^2 - \omega_0^2) \tilde{\varphi}(p,t)$$

$$\tilde{\varphi}(p,t) = A_p \cos(\omega t + \phi_p) \qquad \omega = \sqrt{\omega_0^2 + p^2}$$

A_p, ϕ_p found from in. conditions

Fancy example: N scalar fields $\underline{\varphi_1}, \underline{\varphi_2}, \ldots \underline{\varphi_N}$ $\quad \underline{\underline{\varphi_a}}$

a index $1, \ldots, N$. Same summation conventions.

$\mathcal{L}(\underline{\varphi_a}, \underline{\underline{\partial_\mu}}\varphi_a)$ symm. $\varphi_a \to \mathcal{O}_{ab}\varphi_b$ $\quad \mathcal{O}_{ab}$ orthogonal matrix

$\underline{\mathcal{O}^{-1}_{ab}}$ meaning $\underline{\mathcal{O}^{-1}_{ac}}\,\underline{\mathcal{O}_{cb}} = \delta_{ab} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & & \ddots \end{bmatrix}$

obeys $\quad \mathcal{O}^{-1}_{ab} = \mathcal{O}_{ba} = \mathcal{O}^T_{ab}$

$$\begin{bmatrix} \cos\boxed{\theta} & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

$$\mathcal{O}^T\mathcal{O} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2+s^2 & cs-sc \\ cs-sc & c^2+s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3×3: Rotation matrices

4×4: $\quad$ " $\quad$ in 4-Dim Space

$N\times N$: $\dfrac{N(N-1)}{2}$ indep. θ's

Symm holds iff every index appears 2x and is summed.

Scalar $\vec{p}\cdot\vec{p} = p_i p_i$

$\mathcal{L}$ is "scalar" under $O(N)$ symm

$$S = \int d^4x \sum \frac{1}{2} \boxed{\partial_\mu \varphi_a} \partial^\mu \varphi^a - \frac{1}{2}\omega_0^2 \boxed{\varphi_a} \varphi_a - \frac{\lambda}{24\hbar} \varphi_a \varphi_a \varphi_b \varphi_b$$

$\mu = 0123$

$a = 12..N$

$$+ \partial_t \partial_t - \vec{\nabla}\cdot\vec{\nabla}$$

~~NO~~ $\cancel{\sum \varphi_a \varphi_a \varphi_a \varphi_a}$

NO $\cancel{p_x^4 + p_y^4 + p_z^4}$

Euler-Lagrange

$$\partial_\mu \frac{\partial \mathcal{L}[\varphi_a, \partial_\mu \varphi_a]}{\partial(\partial^\mu \varphi_a)} = \frac{\partial \mathcal{L}}{\partial \varphi_a}$$

$$\partial_\mu \partial^\mu \varphi_a = -\omega_0^2 \varphi_a - \frac{\lambda}{6\hbar} \varphi_a \varphi_b \varphi_b$$

Nonlinear. Nasty. Nontrivial.

BUT

$$J^{\mu}_{ab} = \varphi_a \partial^\mu \varphi_b - \varphi_b \partial^\mu \varphi_a$$

$a \neq b$

$a < b$

$\frac{N(N-1)}{2}$ indep. J^{ab}'s.

$\partial_\mu J^\mu_{ab} = 0$ (?? Really ??)

$$= \cancel{(\partial_\mu \varphi_a) \partial^\mu \varphi_b} + \varphi_a \partial_\mu \partial^\mu \varphi_b - \cancel{(\partial_\mu \varphi_b)(\partial^\mu \varphi_a)} - \varphi_b \partial_\mu \partial^\mu \varphi_a$$

$$\partial_\mu \varphi_a \partial^\mu \varphi_b = g^{\mu\nu} \partial_\mu \varphi_a \partial_\nu \varphi_b \qquad - g^{\nu\mu} \partial_\mu \varphi_b \partial_\nu \varphi_a$$

EOM: $\partial_\mu \partial^\mu \varphi_b = -\omega_0^2 \varphi_b - \frac{\lambda}{6t_1} \varphi_b \varphi_c \varphi_c$ same for φ_a

$$\partial_\mu J^\mu_{ab} = \varphi_a \cancel{(-\omega_0^2) \varphi_b} - \cancel{\frac{\lambda}{6t_1} \varphi_a \varphi_b \varphi_c \varphi_c} + \cancel{\varphi_b \omega_0^2 \varphi_a} + \cancel{\frac{\lambda}{6t_1} \varphi_b \varphi_a \varphi_c \varphi_c}$$

$= 0$

Nöther's Theorem (Emmy Nöther)

$\varphi_a \quad a = 1 \ldots N \qquad \forall a > b, \qquad \varphi_a \to \varphi_a + \epsilon \varphi_b \qquad \epsilon$ infinitesimal

$\varphi_b \to \varphi_b - \epsilon \varphi_a$

$$\begin{bmatrix} 1 & \epsilon \\ \underline{-\epsilon} & 1 \end{bmatrix} \begin{pmatrix} \varphi_a \\ \varphi_b \end{pmatrix}$$

$\mathcal{L}$ is unchg'd.

$$T_1^{ab} = \begin{array}{c} \scriptstyle 12 \\ \begin{bmatrix} 0 & -i & 0 & \\ i & 0 & 0 & \cdots \\ 0 & 0 & 0 & \\ & \vdots & & \end{bmatrix} \end{array}$$

Most general $\underline{\varphi_a} \to \underline{\varepsilon_A}\, i\, \underline{T_A^{ab}}\, \varphi_b$

herm. Matrices

a, b row & column

A index on

how many indep. matrices exist

$$T_2^{ab} = \begin{array}{c} \scriptstyle 13 \\ \begin{bmatrix} 0 & 0 & -i & \\ 0 & 0 & 0 & \cdots \\ +i & 0 & 0 & \\ & \vdots & & \end{bmatrix} \end{array}$$

$$T_3^{ab} = \begin{bmatrix} 0 & 0 & 0 & \\ 0 & 0 & -i & \cdots \\ 0 & i & 0 & \\ & \vdots & & \end{bmatrix}$$

$\underline{a}$: $1, 2, \ldots N$

$\underline{A}$: all pairs $a > b$: $\frac{N(N-1)}{2}$ choices

ε_A are "angles"

T_A^{ab} are generators

of transformations (Rotations in space of fields - not in phys. space)

$$\partial_\mu \varphi_a \longrightarrow \partial_\mu \bar{\varphi}_a = \partial_\mu \varphi_a + i\,\epsilon_A T_A^{ab}\, \partial_\mu \varphi_b$$

$$\mathcal{L}[\varphi_a] \longrightarrow \mathcal{L}[\bar{\varphi}_a] = \mathcal{L}[\varphi_a]$$ value is unchg'd

not now

OR AT WORST $$\mathcal{L}[\bar{\varphi}_a] = \mathcal{L}[\varphi_a] + \epsilon_A\, \partial_\mu J^\mu_A(\varphi)$$

Total Deriv. extra stuff

$$0 = \mathcal{L}[\bar{\varphi}_a] - \mathcal{L}[\varphi_a] - \epsilon_A\, \partial_\mu J^\mu_A$$

Expand to 1-order in ϵ_A.

$$\mathcal{L}[\bar{\varphi}_a, \partial_\mu \bar{\varphi}_a]$$

Π^μ

$$0 = \delta\bar{\varphi}_a \frac{\partial \mathcal{L}}{\partial \bar{\varphi}_a} + \delta\,\partial_\mu \bar{\varphi}_a \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \bar{\varphi}_a} + \cancel{\mathcal{L}[\bar{\varphi}_a]} - \cancel{\mathcal{L}[\varphi_a]} - \epsilon_A\, \partial_\mu J^\mu_A$$

$$0 = \delta\bar{\varphi}_a \frac{\partial \mathcal{L}}{\partial \varphi_a} + \delta \partial_\mu \varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} - \epsilon_A \partial_\mu J^\mu_A$$

$$\delta\bar{\varphi}_a = i\underline{\epsilon_A T_A^{ab}}\varphi_b$$

$$i\epsilon_A T_A^{ab}\left[\varphi_b \cancel{\frac{\partial \mathcal{L}}{\partial \varphi_a}} + (\partial_\mu \varphi_b)\pi^\mu_a\right] - \epsilon_A \partial_\mu J^\mu_A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_a} = \partial_\mu \pi^\mu_a \quad \text{EL eq.}$$

$$= \partial_\mu\left[\varphi_b \pi^\mu_a\right] - \cancel{\varphi_b \partial_\mu \pi^\mu_a}$$

Nöther current J^μ_A

$$\epsilon_A\left\{ i T_A^{ab}\partial_\mu(\varphi_b \pi^\mu_a) - \underline{\partial_\mu} J^\mu_A \right\} = 0$$

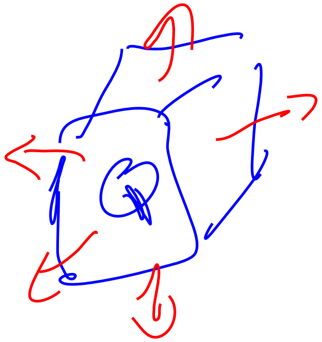
or

$$\epsilon_A \partial_\mu \overbrace{\left[i T_A^{ab}\varphi_b \pi^\mu_a - J^\mu_A\right]}^{J^\mu_A} = 0$$

$$\epsilon_A \boxed{\partial_\mu J^\mu_A = 0} \quad \text{any } \epsilon_A$$

Useful? Yes

$$\partial_t J^0 = -\vec{\nabla}\cdot\vec{J}$$



$$\frac{\partial Q}{\partial t} = -\oint d\vec{\Sigma}\cdot\vec{J}_Q$$

$$\partial_\mu J^\mu = 0$$

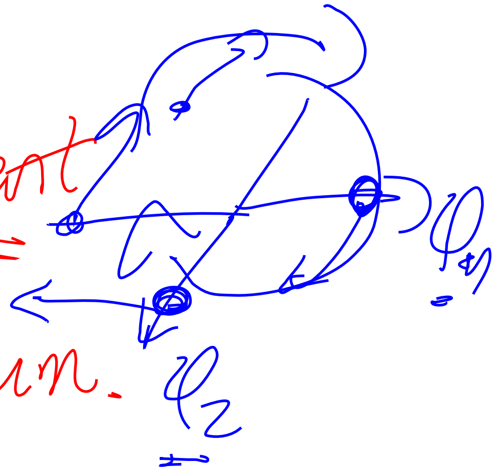
$$Q = \int d^3x\, J^0$$

All space:

$$\boxed{Q \equiv \int d^3x\, J^0 \quad \text{Yes}}$$

$$\frac{d}{dt}Q = \int d^3x\, \frac{\partial J^0}{\partial t} = -\int d^3x\, \vec{\nabla}\cdot\vec{J} = \cancel{\oint d\vec{\Sigma}\cdot\vec{J}} = 0$$

$$Q_{ab} = \int d^3x\left(-\varphi_a\dot{\varphi}_b + \dot{\varphi}_a\varphi_b\right)$$ is constant



field-space angular momentum.

What about actual angular momentum?

$E, \vec{P}, \vec{J}$ or $\vec{L}$ ... Conserved things, what about them?

$$\vec{x} \to \vec{x} + \vec{\xi}$$

$\vec{\xi}$ translation of coord's

$$\varphi(\vec{x}) \to \varphi(\vec{x}+\vec{\xi}) = \varphi(x) + \xi^\mu \partial_\mu \varphi(x) + \mathcal{O}(\xi^2)$$

$$\mathcal{L}(\varphi(x+\xi)) = \mathcal{L}(\varphi(x)) + \xi^\mu \partial_\mu \mathcal{L}[\varphi(x)]$$

$\xi^\mu : \epsilon_A$ Like A index: which trans. I discuss

$\boxed{\mathcal{L}}$ $\{\mathcal{P}\}$

Start again here on Friday

$\partial_\mu J^{\mu\nu} = 0$ $T^{\mu\nu}$ stress tensor

$$\int d^3x \, J^{0\,\begin{smallmatrix}0\\1\\2\\3\end{smallmatrix}} = \begin{pmatrix} \to E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$