Reminder

Recordings (2 books)

lecture notes

lectures

lecture slides (after)

Recordings ⇐ PW prot.

HW

Solutions ⇐ PW prot.

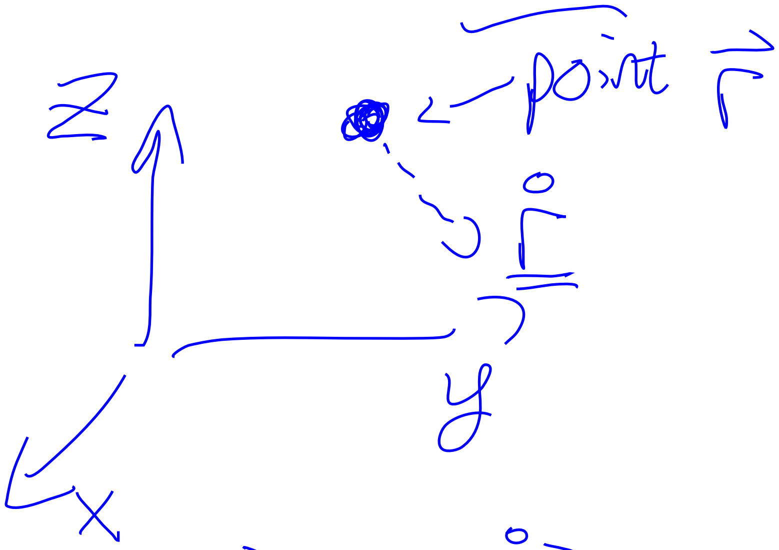
1: lectures & HW solutions are password protected. See Chat!!

2: Wednesday help

↳ See Chat.

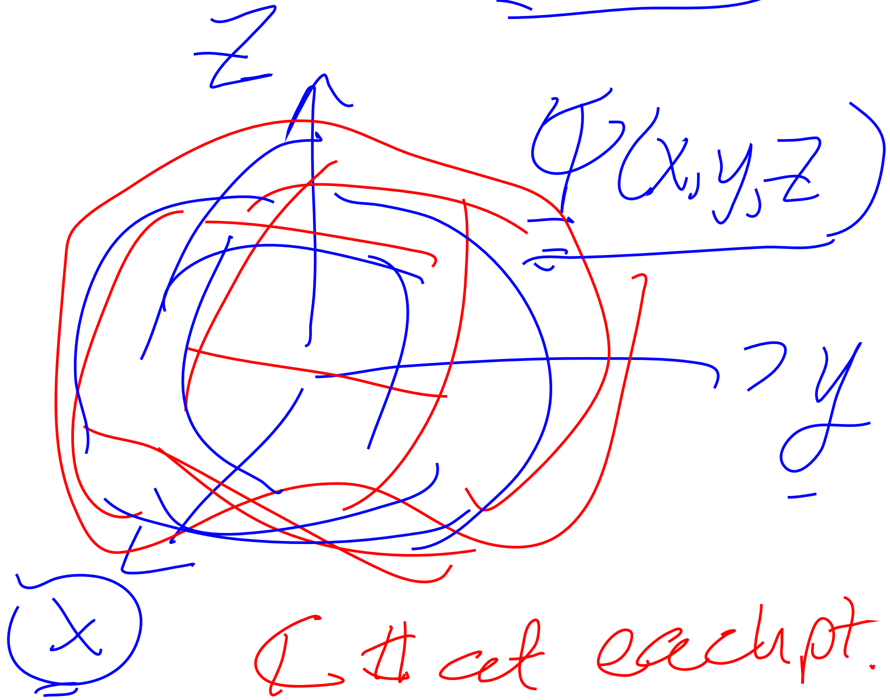
Quantum Field Theory

Classical



$\vec{p} = m\dot{\vec{r}}$

Quantum Mech



ψ at each pt. in sp.

Mom: operator $\hat{p}_j = -i\dfrac{\partial}{\partial x_j}$

$\hat{p}\;\psi$ as op.

$\hat{p}$ info. is already stored in ψ

Classical Field Thy

State $\varphi(x)$ value at each pt in space

Canonmom $\pi(x) = \dot{\varphi}(x)$

how fast φ is changing

Kronecker

$\frac{\partial}{\partial x_i} x_j = \delta_{ij}$

Dirac

$\frac{\delta}{\delta \varphi(x)} \varphi(y) = \delta(x-y)$

C functional

Quantum Field Thy.

State-space: all possible $\varphi(x)$ choices

$\Psi[\varphi[x]]$ Every poss. class field config, C#

$\pi(x) = -i \frac{\delta}{\delta \varphi(x)} \Psi[\varphi(y)]$

Ψ is a function over functions in Contin space

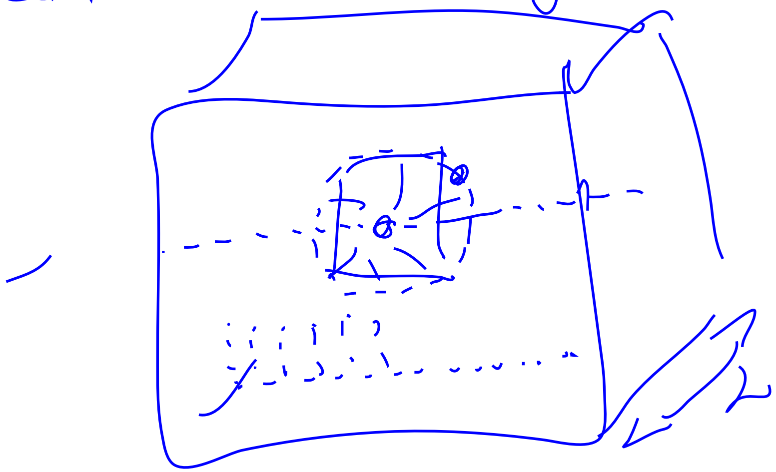
2nd Quantization

Ask a Mathematician: Does this make sense?
Answer: no. Or we don't think so.

But: I took a crystal lattice: finite # of DOF (atoms)
Built a long-dist descrip. → Field Thy
Pretend contin space - and it worked.

Shortest probed dist? Highest E: 10 TeV → 10^{-20} m.
the

Assume space discrete $\boxed{10^{-24}\text{ m}}$ (really: 10^{-35} m.)
(Regularization Regulation)



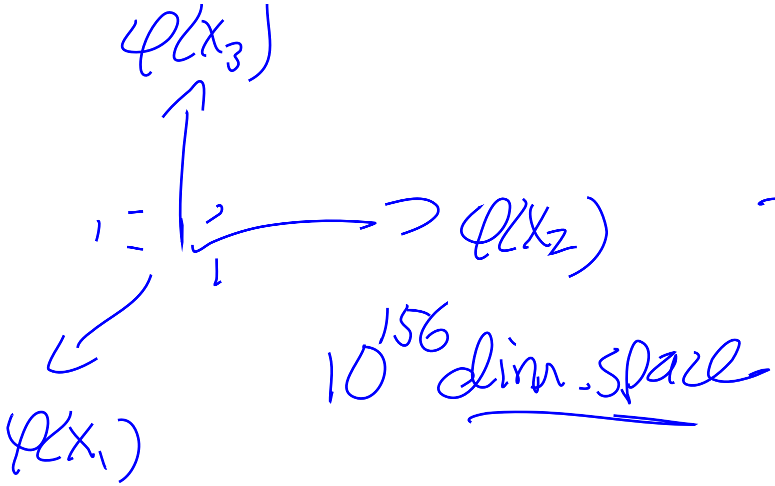
box, L×L×L   L ≥ 10^{12} lightyr
$\boxed{= 10^{28}\text{ m}}$

No consequence

10^{52} points across latt.

Grid with $(10^{52})^3 = 10^{156}$ points (or more)

$\varphi(x) = (\varphi(x_1), \varphi(x_2), \varphi(x_3), \ldots \varphi(x_{10^{156}}))$.



Ψ - wave funct. on 10^{156} dim space.

Now it's well defined

Size L: $L/a > 10^{52}$ ....

Spacing a

How do answers depend on a?

Take $\lim a \to 0$. Version 3 - real version

To define thy, need "cutoff" removing arb. short dist.
Hope/need/expect $a \to 0$ lim. works/exists.

Totally tricky & nontrivial Renorm. Group.

Simplest Theory: 1 real scalar field.

$\varphi(x)$ func. over $\vec{x}$ in 3+1 Dim (class.)

|| $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2$

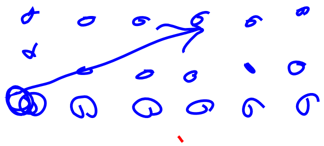
NO φ, φ^3, φ^4 today! (can make $m^2=0$)

$$L = \int d^3x\, \mathcal{L} \longrightarrow a^3 \sum_{\substack{n_1\\ n_2 \\ n_3}\in\mathbb{Z}} \left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2\right]$$

integers

huh?? $\partial_0\varphi = \dot{\varphi}$ is OK

$$\vec{\nabla}_i\varphi = \frac{\varphi(x+a\hat{i})-\varphi(x)}{a}$$



$\vec{x} = a\vec{n}$

vector of int's

space latt

time continuous

What is $\varphi(x)$?? An operator (like x in QM)

$\Psi(\varphi(x))$ or $|\Psi\rangle$

$\boxed{\langle\bar{\varphi}|}\,\hat{\varphi}(x)|\Psi\rangle = \bar{\varphi}(x)\langle\bar{\varphi}|\Psi\rangle$

state where $\Psi=0$ except on specific class. config. where φ = func.

$$L = a^3 \sum_{\vec n \in \mathbb{Z}^3} \frac{1}{2}\dot\varphi^2(x) - \frac{|\nabla\varphi|^2(x)}{2} - \frac{m^2}{2}\varphi^2(x)$$

$\pi^2(x)$ canon momentum ?? (class. π^0)

$\pi(x)$ also an op. class.

$$\pi(\vec x) = \frac{\partial L}{\partial\, \partial_0\varphi(\vec x)} = a^3\, \partial_0\varphi(\vec x), \qquad \vec x = a\vec n$$

$$H = \sum_n \pi\, \partial_0\varphi - L = \sum_n \left[a^{-3}\frac{1}{2}\pi(x)\pi(x) + \frac{a^3}{2}|\nabla\varphi|^2 + \frac{a^3 m^2}{2}\varphi^2(x) \right], \qquad \vec x = a\vec n$$

$$\boxed{\hat\pi(\vec x) = -i\frac{\partial}{\partial\varphi(x)}}$$

in "standard" "coord" basis

$$\Psi(\bar\varphi(x)) = \langle \bar{\vec\varphi}(x) | \Psi \rangle$$

$$\langle \bar\varphi(x) | \pi(x) | \Psi \rangle = -i\frac{\partial}{\partial\bar\varphi(x)}\Psi(\bar\varphi(x))$$

Or summarize relation between Q, mom P

$\hat{p}\,\psi(Q) = -i \frac{\partial}{\partial Q}\psi(Q)$ QP vs PQ ....

$[P_i, Q_j] = -i\,\delta_{ij}$ expect same

Commutation: $[\hat{\pi}_{\vec{n}}, \hat{\varphi}_{\vec{m}}] = -i\,\delta_{\vec{n},\vec{m}}$ or: $[\pi(\vec{x}), \varphi(\vec{y})] = -i\,\delta_{xy}$

10^{156} diff $\vec{m}$ index val.

Kronecker

$\pi(\vec{x}) = a^{-3}\pi(x)$ not true canon mom.

$[\hat{\pi}(x), \hat{\varphi}(y)] = -i\, \underbrace{a^{-3}\delta_{xy}}_{\text{Dirac delta}}$

$\underbrace{a^{3}\sum_{\vec{n}}}_{\int d^3x} \underbrace{a^{-3}\delta_{nm}}_{\delta^3(x-y)} = 1$

$$H = \int d^3x \left[\frac{1}{2}\hat{\pi}^2(x) + \frac{1}{2}(\nabla\hat{\varphi})^2(x) + \frac{m^2}{2}\hat{\varphi}^2(x) \right]$$

" $\quad a^3 \sum_{\substack{x=\vec{n}a \\ n\in\mathbb{Z}^3}}$

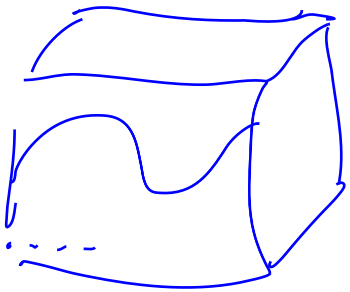
$$([\pi(x), \varphi(y)] = -i\delta^3(x-y)$$

H not diag in (φ, π) basis because $\boxed{\nabla_i\varphi(x)} = \frac{\varphi(x+ae_i)-\varphi(x)}{a}$ $\quad \to -ip$

$p \to -p$

Fourier techniques

$$\tilde{\varphi}^*(p) = \tilde{\varphi}(-p) \qquad \tilde{\varphi}(p) = \int d^3x\, e^{+i\vec{p}\cdot\vec{x}}\varphi(x) \qquad p = \vec{n}\frac{2\pi}{L}$$



$$\varphi(x) = \sum L^{-3}\tilde{\varphi}(p)\, e^{+i\vec{p}\cdot\vec{x}}$$

$p = \vec{n}\frac{2\pi}{L}$

L, a finite

$\tilde{\pi}(p)$, similarly.

$$\vec{n}_{max} = \frac{L}{2a}$$

$\tilde{\varphi}(p)$ complex while $\varphi(x)$ real

$$\tilde{\varphi}(-p) = \left(\tilde{\varphi}(p)\right)^*$$

$$\int d^3x \, \frac{\pi(x)\pi(x)}{2} = \sum_{p_1 = \vec{n}_1 \frac{2\pi}{L}} \sum_{p_2 = \vec{n}_2 \frac{2\pi}{L}} L^{-6} \underbrace{\int d^3x \, e^{i\vec{p}_1 \cdot x} e^{i\vec{p}_2 \cdot x}}_{L^3 \delta^3_{n_1+n_2} \quad n_2 = -n_1} \underbrace{\frac{\tilde{\pi}(p_1)\tilde{\pi}(p_2)}{2}}_{\text{no } x}$$

$$= \sum_{p = \vec{n}\frac{2\pi}{L}} L^{-3} \underbrace{\frac{\tilde{\pi}(p)\tilde{\pi}^*(p)}{2}}_{= \frac{\tilde{\pi}_r^2(p) + \tilde{\pi}_i^2(p)}{2}}$$

$$\tilde{\pi}(p) = \pi_r(p) + i\pi_i(p)$$

$$H = \sum_{p = \vec{n}\frac{2\pi}{L}} L^{-3} \int \frac{\tilde{\pi}(p)\tilde{\pi}(-p)}{2} + \left(m^2 + \frac{4}{a^2}\sin^2\left(\frac{pa}{2}\right)\right)\tilde{\varphi}(p)\tilde{\varphi}(-p)$$

$$\sum_i \frac{4}{a^2}\sin^2\left(\frac{p_i a}{2}\right) = \vec{p}^{\,2}$$

$$\tilde{\pi}(p)\tilde{\pi}(-p)$$

SHO's? Yes! $\omega_p = \sqrt{p^2 + m^2}$ $\quad \omega_p = |p|$ if $m=0$

$$a_p = \frac{1}{\sqrt{2L^3}}\left(\sqrt{2\omega_p}\,\tilde{\varphi}(p) + \frac{i}{\sqrt{2\omega_p}}\tilde{\pi}(p)\right)$$

note $\tilde{\varphi}^*(-p) = \tilde{\varphi}(p)$

$$a_p^\dagger = \frac{1}{\sqrt{2L^3}}\left(\sqrt{2\omega}\,\tilde{\varphi}(-p) - \frac{i}{\sqrt{2\omega_p}}\tilde{\pi}(-p)\right)$$

really is $*$ of a_p

Much work: $[a_p^\dagger, a_q] = \delta_{p,q}$ kroneker

$$H = \sum_{p=\vec{n}\frac{2\pi}{L}} \left(\frac{a_p a_p^\dagger}{2} + \frac{a_{-p}^\dagger a_{-p}}{2} \right) \omega_p = \sum_{p=\vec{n}\frac{2\pi}{L}} \left(a_p^\dagger a_p + \frac{1}{2} \right) \omega_p$$

10^{156} independent simple harm. osc.

If $a \to 0$ ⇒ can – $\vec{n}$ sum runs over $\mathbb{Z}^3$

no, just a box!

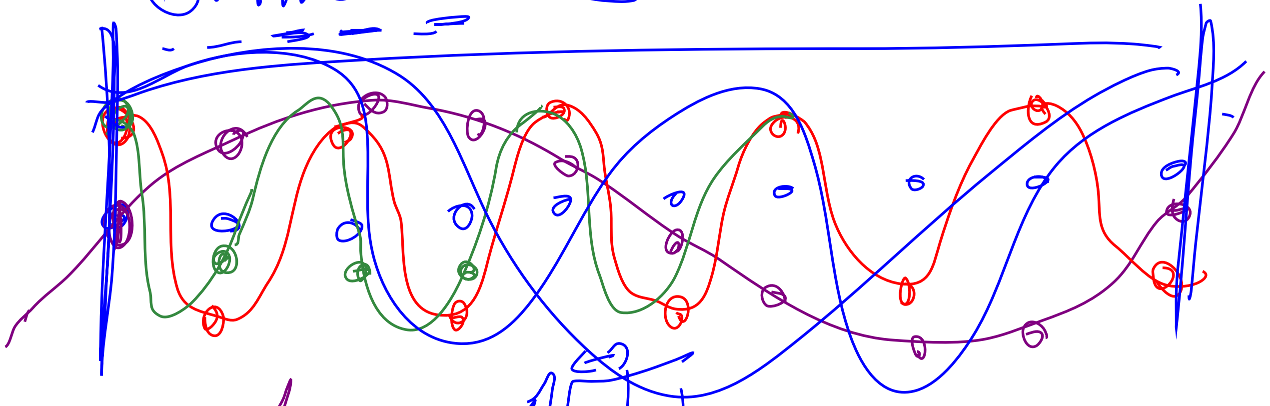
$$\tilde{\varphi}(p) = \sqrt{\frac{L^3}{2\omega_p}} \left(a_p + a_{-p}^\dagger \right)$$

$$\varphi(x) = \sum_{p=n\frac{2\pi}{L}} \frac{\sqrt{\omega_p}}{\sqrt{2\cancel{\omega_p}L^3}} \left(\underbrace{-i e^{i\vec{p}\cdot\vec{x}} a_p}_{} + \underbrace{i e^{-i\vec{p}\cdot\vec{x}} a_p^\dagger}_{} \right)$$

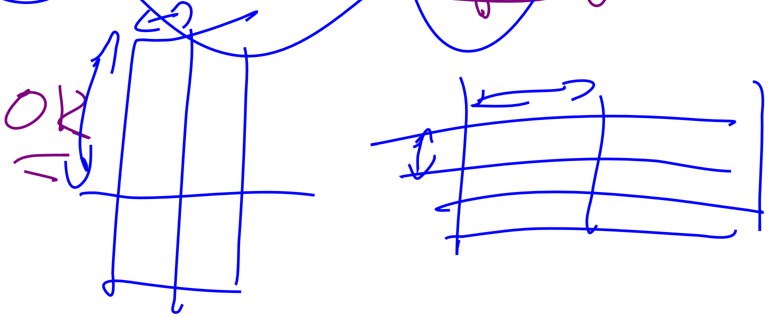
$\pi(x)$

both hermitian

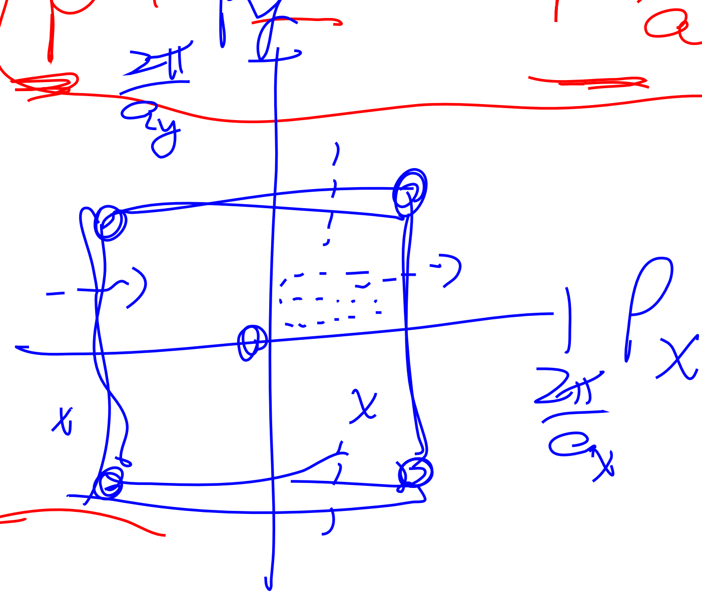
Brillouin Zone

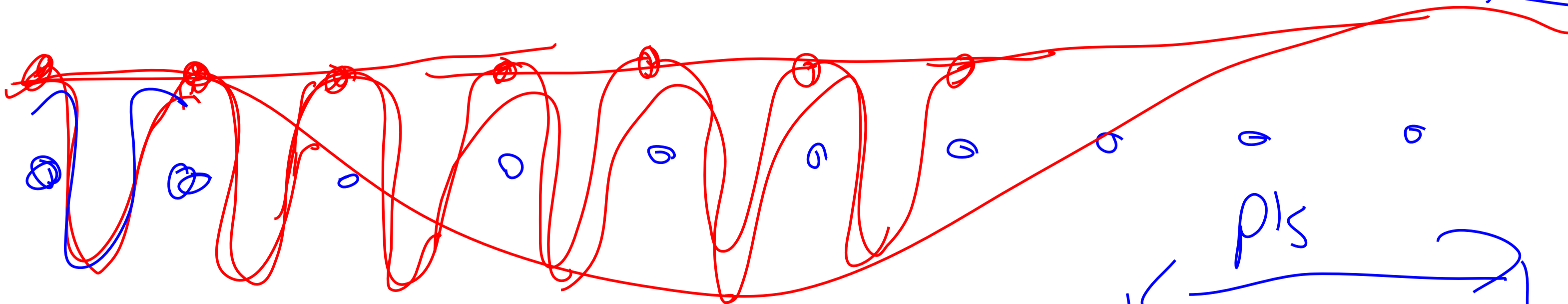


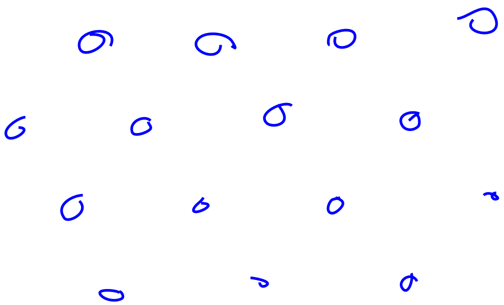
$\vec{p} \ll \frac{1}{a}$ is OK



$\vec{p}$ is same as $p + \frac{2\pi}{a}$







finite box
↳ discrete p's
discrete box
↳ finite p-region