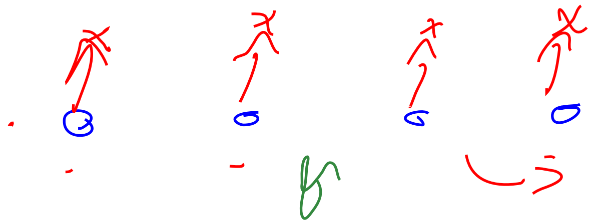
# Lecture 6: more about free scalar field theory

- What are states, really?
- Particles which don't spread across whole universe, and $L \to \infty$ limit
  - ↳ wave packets
- Time evolution
- Two-point function & fluctuations in field.

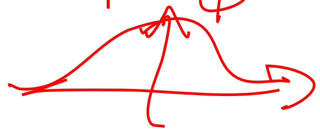
$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2 \cancel{\left[ -\frac{\lambda}{24}\varphi^4 \right]}$$

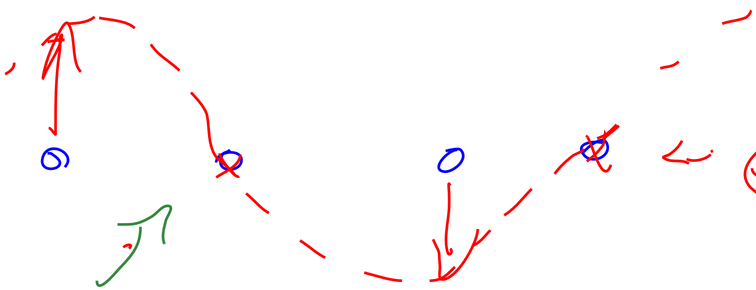
1-Dim space

# 4 Simple HO

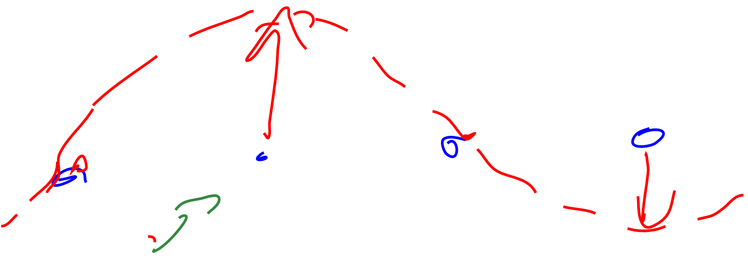


$p=0$ : all move together

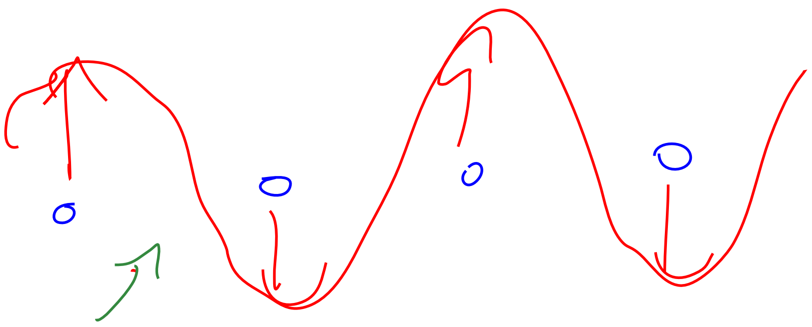


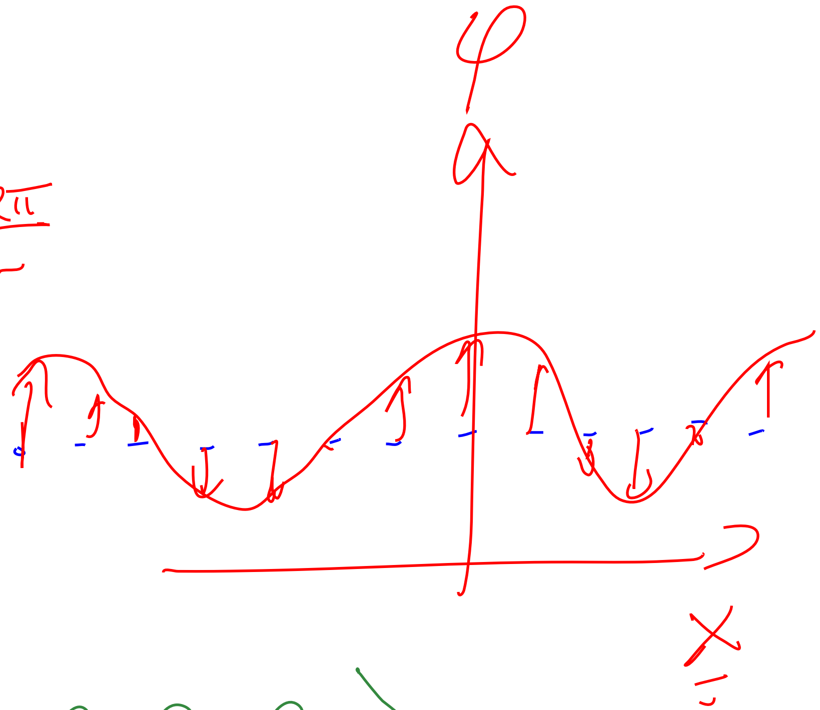
$$\frac{e^{ipx}+e^{-ipx}}{}\qquad p=\pm 1\,\frac{2\pi}{L}$$



$$\frac{e^{ipx}-e^{-ipx}}{}$$



$$p=2\,\frac{2\pi}{L}$$

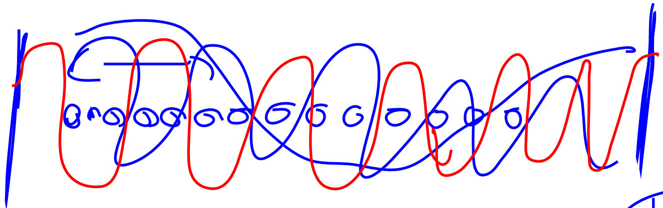



$|n_1\, n_2\, n_3\, n_4\rangle$

$|0\,0\,0\,0\rangle$

$|1\,1\,2\,0\,1\rangle$ etc

Lim as latt spacing $\to 0$



$|n_0\, n_1\, n_2\, n_3 \ldots \rangle$

If I really want $a \to 0$ $\infty$ # of SHO's.

1) Is that OK?

2) What's basis?

$|n_0\, n_1\, n_2 \ldots \rangle$ n's are non-neg int's.

Countable basis.

And $\sum n_i$ total occ. of all states is finite ← Fock Space: Hilbert sp. w. countable dimensionality (separable) OK

OR you let an $\infty$ # of n's be nonzero.

uncountable basis

uncountable basis of indep. states.

Uncountable, non-sep Hilbert sp → QM is toast

What is a particle?

It's a SHO being in $|1\rangle$.

or $c_1|\underline{100}\rangle + c_2|\underline{010}\rangle + c_3|\underline{001}\rangle$

lin comb. of states where $\sum_i n_i = 1$

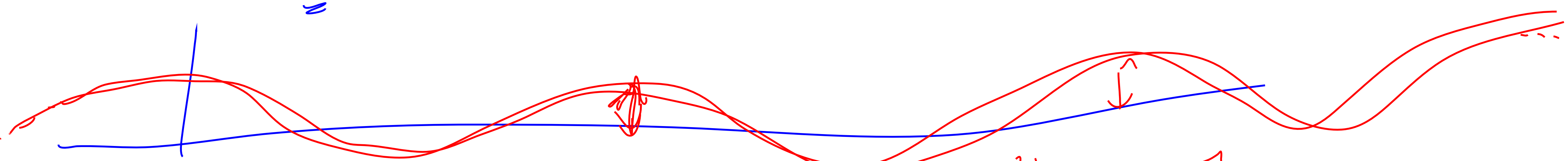
1-part state

2-part state

$c_{200}|200\rangle + c_{110}|110\rangle + \ldots$

comb of $|n_1 n_2 n_3 \ldots\rangle$ where $\sum_i n_i = \underline{2}$

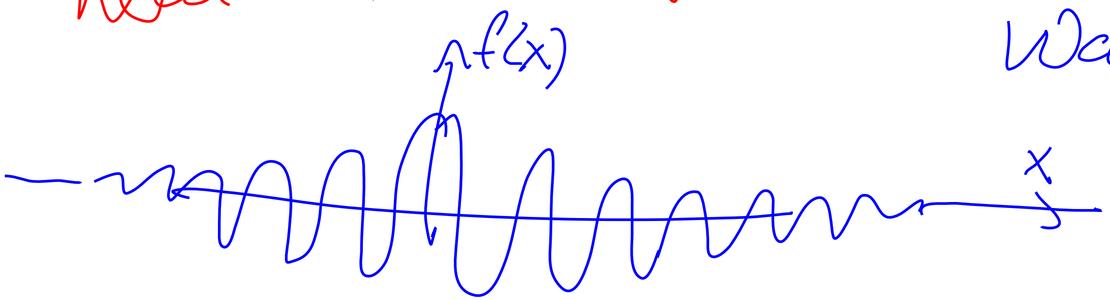
$$\varphi(x) = \sum_{\vec p = \frac{2\pi}{L}\vec n} \left( e^{i\vec p\cdot\vec x} \frac{a_p}{\sqrt{2\omega_p L^3}} + \frac{a_p^\dagger}{\sqrt{2\omega_p L^3}} e^{-i\vec p\cdot\vec x} \right)$$



1 part. spread over whole universe!!

Real "normal" part's are lin. comb. of these

wave packet $A\, e^{i\vec p\cdot\vec x}\, e^{-x^2/2\sigma^2}$




Approx pos $x=0$
mom $p$

$$f(x) \longrightarrow \tilde f(p) = \int d^3x\, e^{-i\vec p\cdot\vec x} f(x)$$

$$|f\rangle = \sum_{\vec p = \frac{2\pi}{L}\vec n} \tilde f(p)\, a_p^\dagger |0\rangle$$

sensible 1-part. state

Normalization?

$$1 = \langle f | f \rangle = \underbrace{\langle 0 | \sum_{p'} a_{p'} \tilde{f}^*(p')}_{\langle f|} \sum_{p} a_p^\dagger \tilde{f}(p) |0\rangle$$

$$\boxed{a_p, a_p^\dagger = \delta_{pp'} + \cancel{a^\dagger a}}$$

$$1 = \langle f | f \rangle = \sum_{p \in \frac{2\pi}{L}\vec{n}} \tilde{f}^* \tilde{f}(p)$$

$\tilde{f}(p) \to 0$ as $L \to \infty$

$$a^3 \sum_{x = a\vec{n}} \to \int d^3x$$

poor norm. conventions

$$\boxed{L^{-3} \sum_{p = \frac{2\pi}{L}\vec{n}}} \to \int \frac{d^3p}{(2\pi)^3}$$

often written $\int \cancel{d}^3 p$

$$\hbar = \frac{h}{2\pi} \qquad \cancel{d}p = \frac{dp}{2\pi} \ \ldots$$

## Old Convention

$$a_p = \frac{1}{\sqrt{2\omega_p L^3}} \int d^3x \, e^{-i\vec{p}\cdot\vec{x}} \left(\omega_p \phi + i\Pi(x)\right)$$

$$|f\rangle = \sum_{p = \frac{2\pi}{L}\vec{n}} \tilde{f}(p)\, a_p^\dagger |0\rangle$$

$$\sum_p \tilde{f}^*\tilde{f}(p) = 1$$

$$[a_p, a_q^\dagger] = \delta^{\text{kron.}}_{p,q}$$

Good: $\| a_p^\dagger |0\rangle \|^2 = 1.$

Bad: $f(x) \propto L^{-3/2}$ yuck...

$\langle 0 | a_p a_p^\dagger | 0\rangle$

## New Convention

$$a_p = \frac{1}{\sqrt{2\omega_p}} \int d^3x \, e^{-i\vec{p}\cdot\vec{x}} \left(\omega_p \phi(x) + i\Pi(x)\right)$$

$$= L^{3/2} a_{\text{old},p}$$

$$|f\rangle = \sum_p L^{-3} \tilde{f}_{\text{new}}\, a^\dagger_{\text{new},p} |0\rangle$$

$$\tilde{f}_{\text{new}}(p) = L^{3/2} \tilde{f}_{\text{old}}(p)$$

fixed as $L \to \infty$

$$\underbrace{\sum_p L^{-3}}\tilde{f}^*\tilde{f}(p) = 1 = \underbrace{\int \frac{d^3p}{(2\pi)^3}}\tilde{f}^*\tilde{f}(p) = 1$$

$$\to [a_p, a_q^\dagger] = L^3 \delta^{\text{kron}}_{p,q}$$

$$\underset{\text{dw.}}{\| a_p^\dagger |0\rangle \|^2} \sim L^3$$

$$= \underbrace{(2\pi)^3 \delta^3(\vec{p} - \vec{q})}_{\text{Dirac}}$$

One more small change:

$$a_{newer,p} = \sqrt{2\omega_p}\, a_{new,p} = \sqrt{2\omega_p}\, L^{3/2} a_{old,p}$$

not Lorentz invariant

$$\varphi(x) = \boxed{\int \frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_p}} \left[ a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right]$$

better than $\int \frac{d^3p}{(2\pi)^3}$

$$[a_p, a_{p'}^\dagger] = \boxed{2\omega_p (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')}$$

$(|p\rangle = a_p^\dagger |0\rangle)$

$$|f\rangle = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \hat{f}_p |p\rangle$$

$$\boxed{\int \frac{d^3p}{(2\pi)^3 2\omega_p}} \hat{f}^*(p)\hat{f}(p) = 1$$

Lorentz-invariant

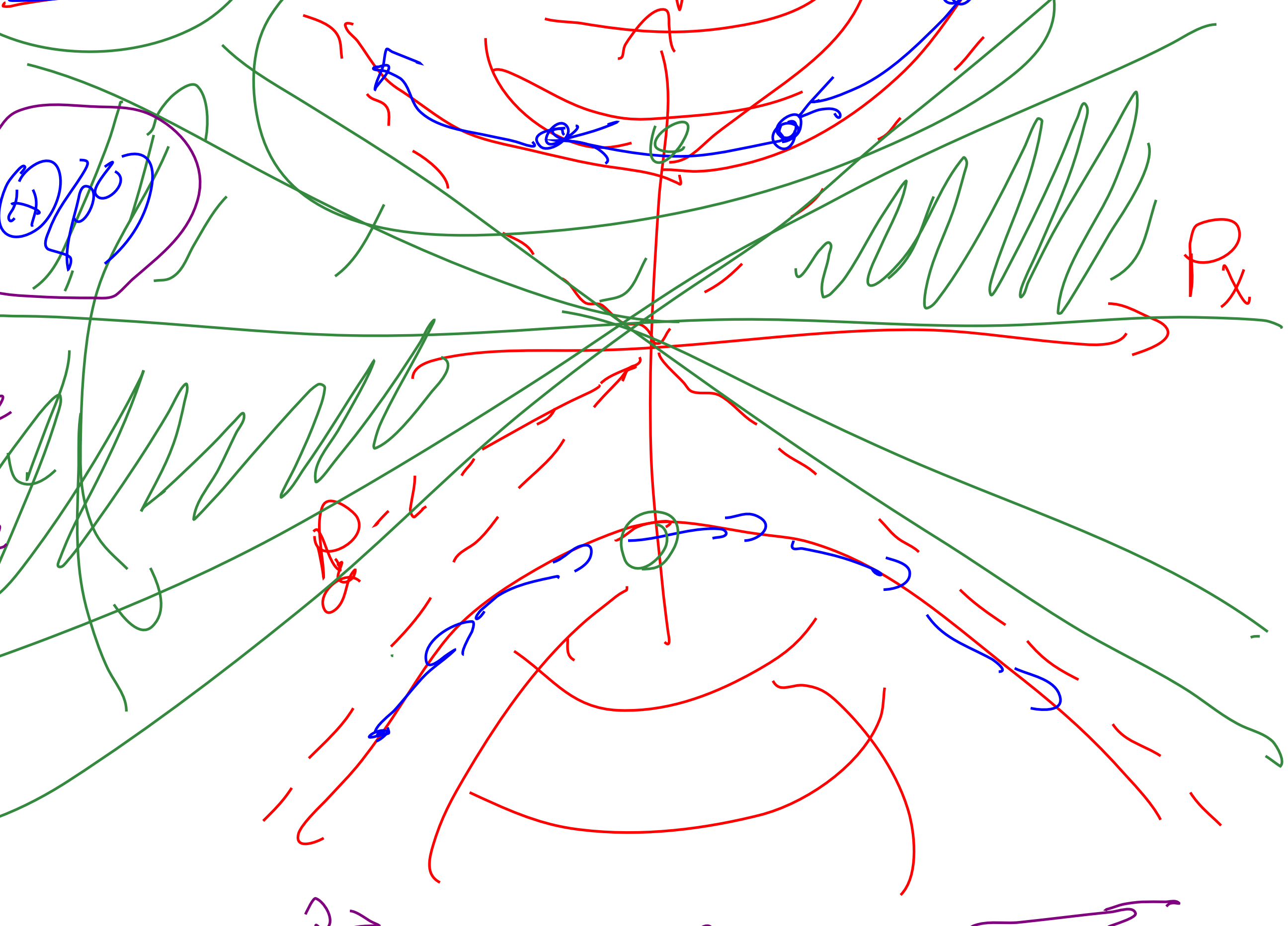
Peskin doesn't do this - or at least takes his time

Srednicki does do this.

Why?

$$\int \frac{d^4 p}{(2\pi)^4} (2\pi)\,\delta(p^2-m^2)\,\Theta(p^0)$$

$$\int \frac{d^3 p}{(2\pi)^3} \int \frac{dp^0}{\cancel{2\pi}}\, \cancel{(2\pi)}\, \delta(p_0^2-\vec{p}^{\,2}-m^2)\,\Theta(p^0)$$




$p_0^2=\vec{p}^{\,2}+m^2$ for $p^0=+\sqrt{\vec{p}^{\,2}+m^2}$

$p^0=-\sqrt{\vec{p}^{\,2}+m^2}$

$p^0=\omega_p$ or $-\omega_p$

$p^0=\omega_p$ and $\boxed{J \propto \frac{1}{2\omega_p}}$

$$\int \frac{d^4 p}{(2\pi)^4}\, 2\pi\,\delta(p^2-m^2)\,\Theta(p^0) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_p}$$

$$p^0 \equiv \omega_p = \sqrt{\vec{p}^{\,2}+m^2}$$

Look at $\int_0^\infty dp^0 \, \delta(p_0^2 - \omega_p^2) \, \cancel{\Theta(p^0)} \; \cancel{= 1 \, ???}$

$$\int_{-\infty}^{+\infty} dx \, \delta(x) = 1$$

change var to $p_0^2$

$y = p_0^2$

$dy = 2p_0 \, dp_0$

$dp_0 = \frac{dy}{2p_0} = \frac{dy}{2\sqrt{y}}$

$$\int_0^\infty \frac{dy}{\boxed{2\sqrt{y}}} \, \delta(y - \omega_p^2) = \frac{1}{2\sqrt{\omega_p^2}} = \frac{1}{2\omega_p}$$

$y = \omega_p^2$

$$\int dx \, g(x) \, \delta(x-y) = g(y)$$

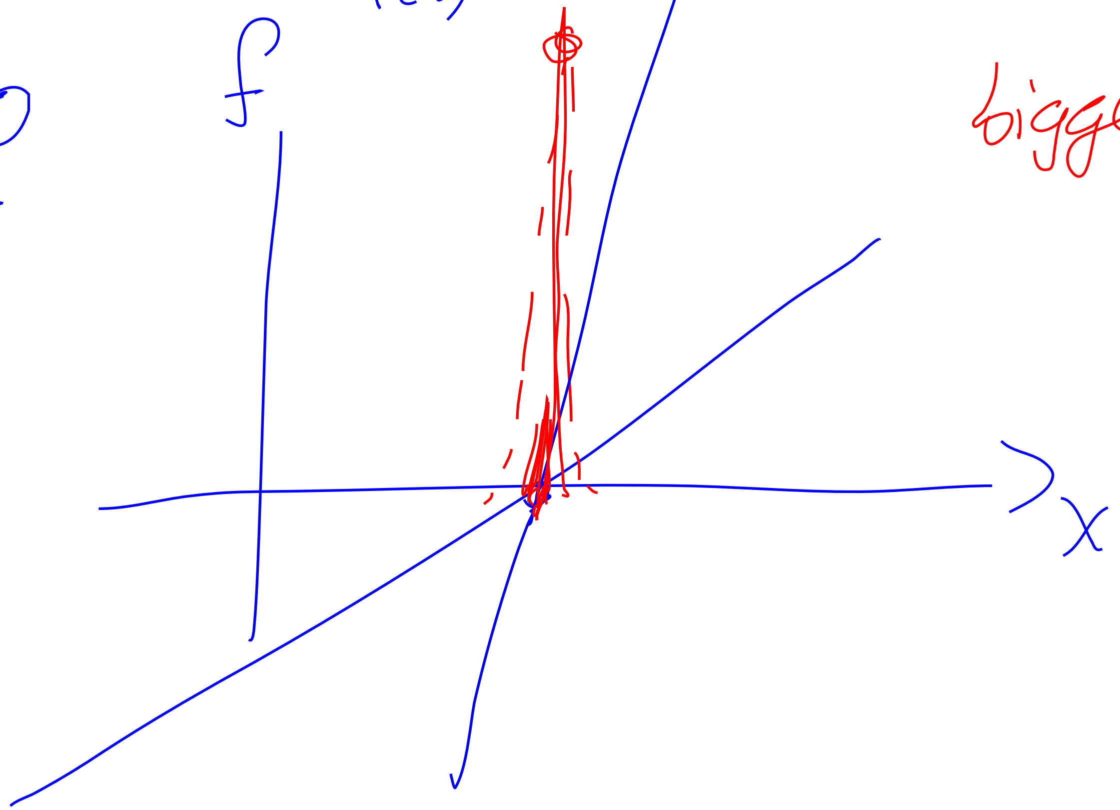
cont. y

$$\int_a^b dx\, \delta(f(x)-y) = \int_{f(a)}^{f(b)} \frac{dx}{d f(x)}\, d f(x)\, \delta(f(x)-y)$$

$a = a \geq 0$

$$\int_{f(a)}^{f(b)} \frac{1}{f'}\, d f(x)\, \delta(f(x)-y) = \frac{1}{\left| f'(x : f(x) = y) \right|}$$

But if $f' \leq 0$




bigger $f'$ → narrower spike

smaller answer

$$\frac{1}{|f'|}$$

## Time evolution & correlations

$$\underbrace{\langle s| e^{iHt}}\, \mathcal{O}\, \underbrace{e^{-iHt} |s\rangle}$$

time evol. of $\mathcal{O} \to \mathcal{O}(t)$

Heisenberg picture

### Heisenberg picture

$$[H, a_p] = -\omega_p a_p$$

$$a_p(t) = e^{iHt} a_p e^{-iHt} = e^{-i\omega_p t} a_p \qquad \omega_p = p^0$$

$$p^2 = m^2 \quad \ldots$$

$$\varphi(x,t) = \int \frac{d^3 p}{(2\pi)^3 2p^0} \left[ \underbrace{e^{-ip^0 t}\, e^{+i\vec{p}\cdot\vec{x}}} a_p + e^{+ip^0 t}\, e^{-i\vec{p}\cdot\vec{x}} a_p^\dagger \right]$$

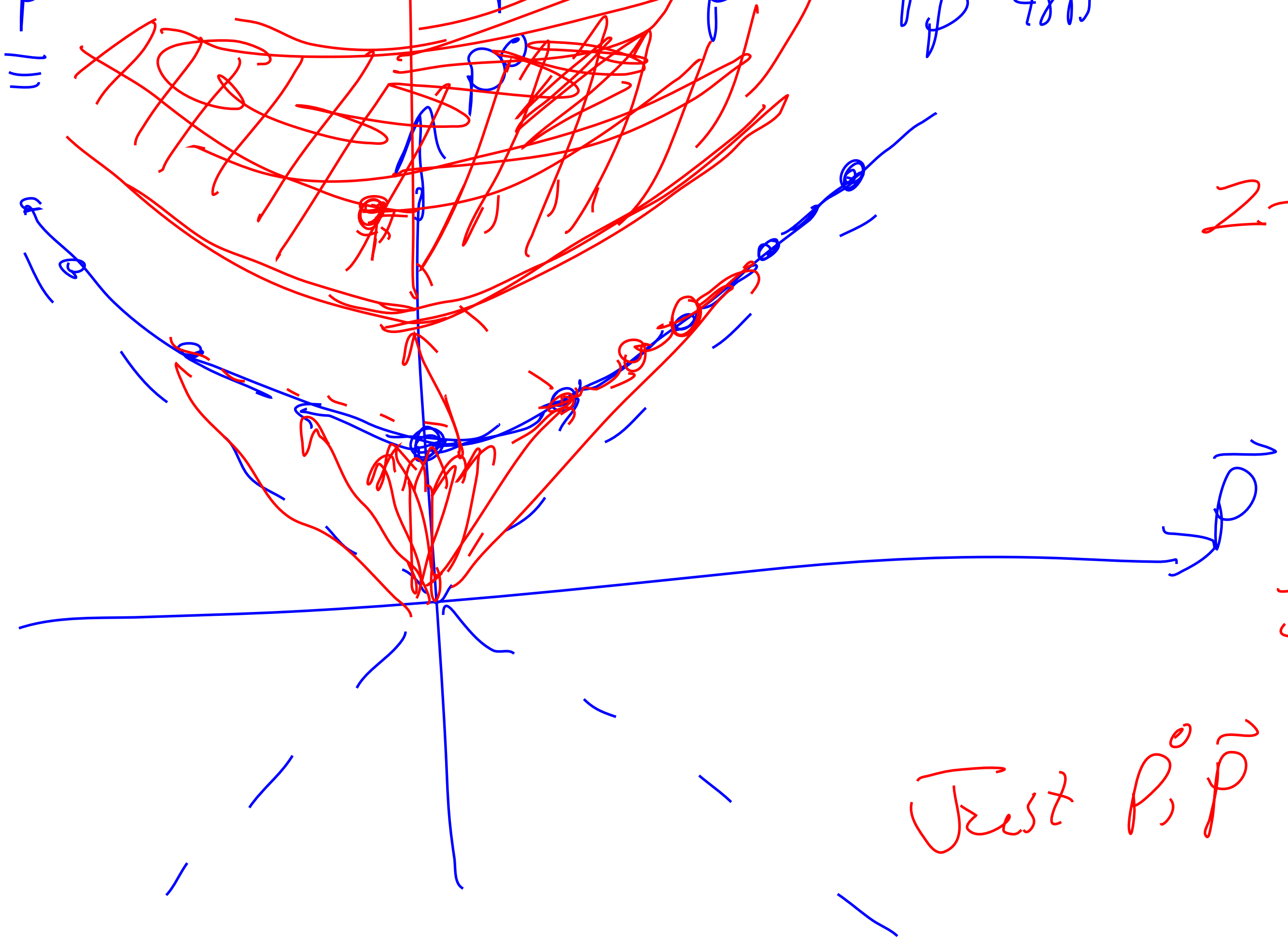
$$e^{-i(p^0 t - \vec{p}\cdot\vec{x})}$$

$$\varphi(x^\mu) = \int \frac{d^4 p}{(2\pi)^4} \left( e^{-ip_\mu x^\mu} a_p + e^{+ip_\mu x^\mu} a_p^\dagger \right) (2\pi)\, \delta(p^2 - m^2)\, \Theta(p^0)$$

Thy has 1-part states — Diff't than

multi-part states —

$p^0$ is related to $\vec{p}$: $p^0 = \sqrt{\vec{p}^2 + m^2}$




1-part has

$$p^0 = \sqrt{\vec{p}^2 + m^2}$$

2-part has

$$p^0 \geq 2\sqrt{\vec{p}^2 + m^2}$$

3part $\geq 3$ . . . .

Just $p^0, \vec{p}$ → 1 part?

or 2+ part's?

$\varphi(x)$ How field fluctuates? $\langle \varphi^2(0) \rangle$ ??

$\langle 0| \varphi(x) \varphi(0) |0\rangle$ x - timelike or spacelike

$= G^{>}_{\varphi\varphi}(x)$ $>$: x is left ver.

$G^{>}(q) = \int d^4x \, e^{iq_\mu x^\mu} G^{>}(x)$ mom space version

Let's do it!

$\varphi(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left( e^{-ip_\mu x^\mu} a_p + e^{+ip_\mu x^\mu} a_p^\dagger \right)$

$\langle 0| \varphi(x)\, \varphi(0) |0\rangle = G^{>}_{\varphi\varphi}(x)$

$0 = \langle 0| a^\dagger$ $\qquad a|0\rangle = 0$

$= \langle 0| \int \frac{d^3p}{(2\pi)^3 2p^0} \left( e^{-ip_\mu x^\mu} a_p + e^{+ip_\mu x^\mu} \cancel{a_p^\dagger} \right) \int \frac{d^3q}{(2\pi)^3 2q^0} \left( \cancel{e\, a_q} + e\, a_q^\dagger \right) |0\rangle$

$0$

$= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 2p^0\, 2q^0} e^{-ip_\mu x^\mu} \underbrace{\langle 0| a_p a_q^\dagger |0\rangle}$

$a_p a_q^\dagger = [a_p, a_q^\dagger] + \cancel{a_q^\dagger a_p}$

$(2\pi)^3 2p^0 \delta^3(\vec{p} - \vec{q})$

$\varphi(0)$ creates part.

$\varphi(x)$ absorbs part.

$$\boxed{G^{>}_{\varphi\varphi}(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip_\mu x^\mu}}$$

$G^{>}(p)$

$G^{>}$ shows how part prop. $0 \to x$

$G^{>}(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu x^\mu} \boxed{2\pi\delta(p^2 - m^2)\, \Theta(p^0)}$

Do the Int! Choose $x^\mu$ spacelike.

Pick frame $x^0 = 0$ $x^3 = x = r$, $x^1, x^2 = 0$ z axis.

$$G(\vec{r}) = \int \frac{d^3p}{\underset{=}{(2\pi)^3}\,\underset{=}{2}\sqrt{\vec{p}^2+m^2}} e^{ip_z r} = \frac{1}{\cancel{16\pi^3}\ 8\pi^2} \int \frac{p^2 dp}{\sqrt{p^2+m^2}} \int_0^{2\pi} d\phi \underbrace{\int_{-1}^{1} d\cos\theta\, e^{ipr\cos\theta}}_{\frac{2}{pr}\sin pr}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty \frac{p \sin(pr)}{\sqrt{p^2+m^2}} dp$$

$$= \frac{m}{4\pi^2 \cancel{r}\, t} \cancel{K_1(mr)}\ H_1(tm)$$

$K_1$: modified Bessel function

$$K_1(x) = \frac{\pi i}{2} e^{\frac{i\pi}{2}} \underbrace{H_1(ix)}_{J_1 + iY_1}$$

Timelike

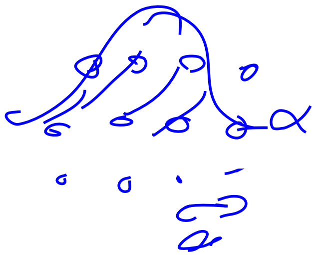
$r \to -ir$

$$\longrightarrow \begin{cases} mr \ll 1 & \boxed{1/4\pi^2 r^2} \\ mr \gg 1 & e^{-mr}\sqrt{m}/2^{5/2}\pi^{3/2} r^{3/2} \end{cases}$$

$$G(r) \xrightarrow{\text{small } r} \frac{1}{4\pi^2 r^2}$$

$$\langle 0|\varphi^2(0)|0\rangle = \lim_{r\to 0} \langle 0|\varphi(r)\varphi(0)|0\rangle = \lim_{r\to 0} \frac{1}{4\pi^2 r^2}$$

Field flucts are big at small dist. $\langle 0|\varphi^2(0)|0\rangle$ doesn't make sense



$\propto \frac{1}{a^2}$

$$\varphi(0) \to \varphi_r(0) = \int \frac{d^3x}{(2\pi r^2)^{3/2}} e^{-x^2/2r^2} \varphi(r)$$

$$\langle \varphi_r^2 \rangle = \int \frac{d^3x\, d^3y}{(2\pi)^3 r^6} e^{-\frac{(x^2+y^2)}{2r^2}} \frac{1}{4\pi^2 |x-y|^2} = \frac{1}{8\pi^2 r^2}$$

$$\frac{1}{8\pi^2 r^2}$$