Free φ theory $\quad \mathcal{L}[\varphi, \partial_\mu \varphi] = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2$ $+E_0$

$|n_1\, n_2\, n_3 \, \ldots\ldots \rangle$ generic

occ. of SHO's $p_1\, p_2$ all possible modes in box

Vacuum $|0\, 0\, 0\, 0\, 0\, 0 \ldots\rangle = |0\rangle$

$|000000\, 1\, 0\, 000\rangle = |p\rangle$ $\quad$ 1 in "p" SHO

which p.

$|00\ldots 1\, 00 \ldots\ldots 1\, 000\rangle = |p_1 p_2\rangle$

$\quad\; p_1 \qquad\qquad p_2$

$$\hat{H}|0\rangle = E_0|0\rangle = \sum_p \Big(\underbrace{a_p^\dagger a_p}_{0} + \frac{1}{2}\Big)\omega_p |0\rangle = \Big(\frac{1}{2}\sum_p \omega_p\Big)|0\rangle$$

$$E_0 \sim L^3 \boxed{\frac{a^{-4}\hbar c}{(2\pi)^4}} \qquad \text{Actual physical value } E_0 : E_0 = L^3 (2.4\,\text{meV})^4$$

Cosmological Const. Problem

I) $\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{24}\varphi^4$

λ pure # (or $\hbar^{-1}$)

$\mathcal{L}(-\varphi) = \mathcal{L}(\varphi) + E_0$

- Is this also a bunch of S.Harm. Osc.? NO

Is there vacuum $|0\rangle$? yes but

II- Are there particles? Probably. See next Q.

- Does this theory exist? It's complicated.

If λ small, exists if $\boxed{a \neq 0}$ even if a is supersuper small. But formal $a \to 0$ limit is a problem:

$\lambda \to 0$ for deep reasons

$\lambda < \frac{4\pi^2}{\log(a^{-1})}$

How to figure this out? $\hat{\varphi}(x)|0\rangle = |\varphi(x)\rangle$

What is this state like?

Ask about amplitudes $\langle\varphi(y)|\varphi(x)\rangle$ $\qquad \cancel{\langle 0|\varphi(x)|0\rangle} = 0$

$$G^{>}(y-x) = \langle 0|\varphi(y)\,\varphi(x)|0\rangle$$

Translation inv:

$$\langle 0|\varphi(y+\xi)\,\varphi(x+\xi)|0\rangle$$
$$= \langle 0|\varphi(y)\,\varphi(x)|0\rangle$$

Pick $\xi = -x$

In free thy, this func. tells us about particles. $G^{>}(p) = 2\pi\,\delta(p^2-m^2)\,\Theta(p^0)$

Shows thy has particles mass m

What about in general?

We have P^μ as conserved currents. $P^\mu = \int d^3x \, T^{\mu 0}(x)$

$\frac{d}{dt} P^\mu = 0$ $\quad [\hat{P}^\mu, \hat{H}] = 0$ $\quad [\hat{P}^\mu, \hat{P}^\nu] = 0$

$\hat{H} = \hat{P}^0$ $\quad$ Translations commute.

P^μ are set of Op's which commute with H & each other.

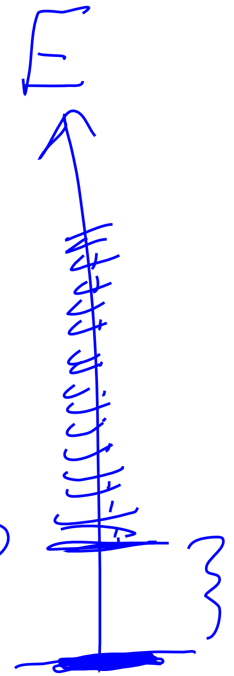
Seek simultaneous eig. states of P^μ.

$\hat{P}^\mu |n\rangle = p_n^\mu |n\rangle$ eigenstates, eig. val. p_n^μ

Basis of states $\mathbb{1} = \sum_n |n\rangle\langle n|$

Choose $P^\mu |0\rangle = 0 |0\rangle$

Assume Mass Gap: Next-smallest E is M, with $M > 0$



Mass Gap

$$G^{>}(y) = \langle 0|\, \underline{\varphi(y)} \sum_n |n\rangle\langle n|\, \varphi(0)\, |0\rangle$$

$$\varphi(\vec{y},t) = e^{+iHt}\, \varphi(\vec{y})\, e^{-iHt} \qquad H = p^0 \text{ gen. of time translations}$$

$$\varphi(\vec{y}) = e^{-i\vec{y}\cdot\vec{p}}\, \varphi(0)\, e^{+i\vec{y}\cdot\vec{p}} \qquad \hat{\vec{p}} \text{ operator gen. of time trans.}$$

$$\varphi(y^\mu) = e^{+i\hat{P}_\mu y^\mu}\, \varphi(0)\, e^{-i\hat{P}_\mu y^\mu}$$

$$G^{>}(y) = \sum_n \langle 0|\, e^{+i\hat{P}^\mu y^\nu g_{\mu\nu}}\, \varphi(0)\, e^{-iP^\alpha y^\beta g_{\alpha\beta}}\, |n\rangle \underbrace{\langle n|\, \varphi(0)\, |0\rangle}$$

$$G^{>}(y) = \sum_n e^{+i0}\, e^{-ip_n^\mu y_\mu}\, \underbrace{\langle 0|\, \varphi(0)\, |n\rangle}_{C_n^*} \underbrace{\langle n|\, \varphi(0)\, |0\rangle}_{C_n} = \sum_n C_n^* C_n\, e^{-ip_n\cdot y}$$

Huh?? Interpret answer??

Write in momentum space

Note for any p_n^μ

$$1 = \int_{m^2}^{\infty} \frac{ds}{2\pi} \int \frac{d^4p}{(2\pi)^4} (2\pi)\,\delta(p^2 - s)\,\Theta(p^0)\,(2\pi)^4 \delta^4(p^\mu - p_n^\mu)$$

$p_n^\mu p_{n\mu} \geq m^2$

$$G^>(y) = \int \frac{ds}{2\pi} \underbrace{\int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu y^\mu}}_{} \underbrace{\Theta(p^0)\, 2\pi\delta(p^2 - s) \underbrace{\sum_n (2\pi)^4 \delta^4(p - p_n) C_n^* C_n}_{= \rho(p^2)}}_{G^>(p)}$$

Depends on p^μ but only through p^2

p^μ and $\Lambda^\mu{}_\nu p^\nu$ gave diff't answers $\rightarrow$ break Lorentz Inv.

$$G^>(y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot y}\, G^>(p) \longrightarrow \text{Free thy: } \underbrace{2\pi\,\delta(p^2-m^2)\,\Theta(p^0)}$$

$$\hookrightarrow \int_{m^2}^{\infty} ds\, \underbrace{\delta(p^2-s)}\,\Theta(p^0)\,\rho(s)$$

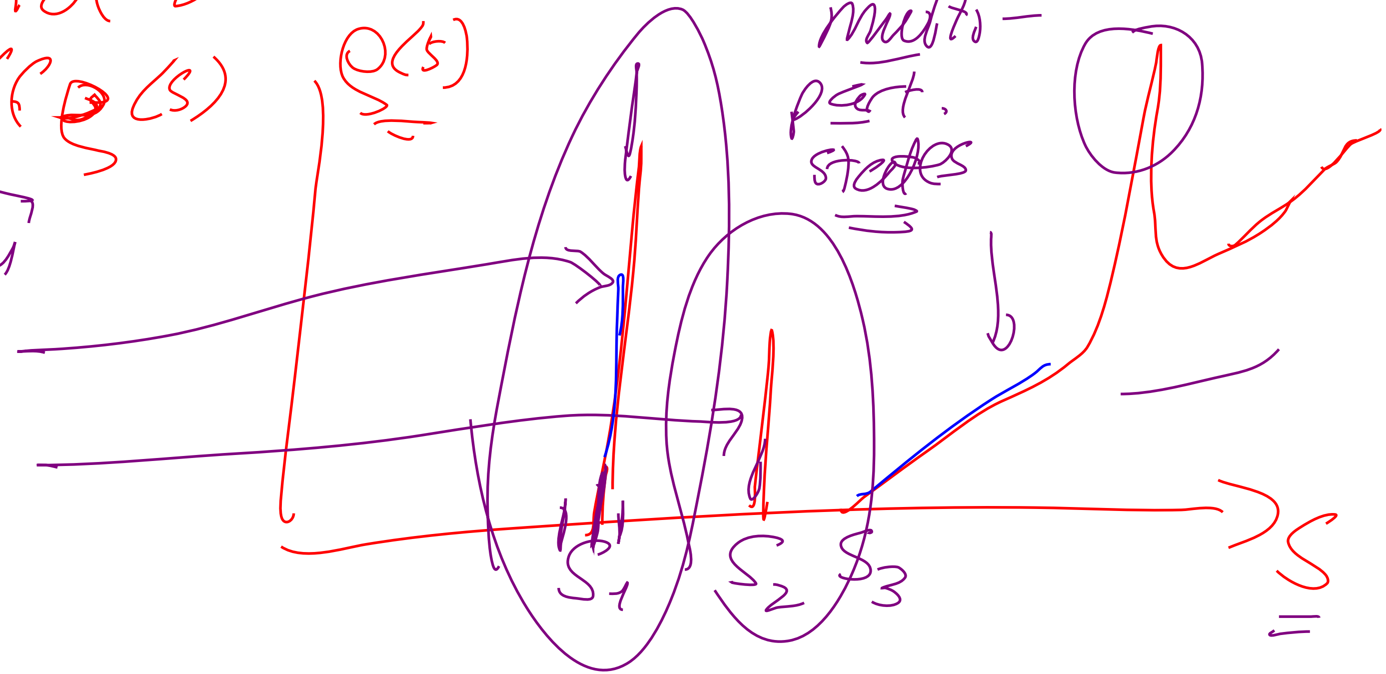
$$= \int \frac{ds}{(2\pi)}\, \rho(s)\, \Delta(x,s)$$

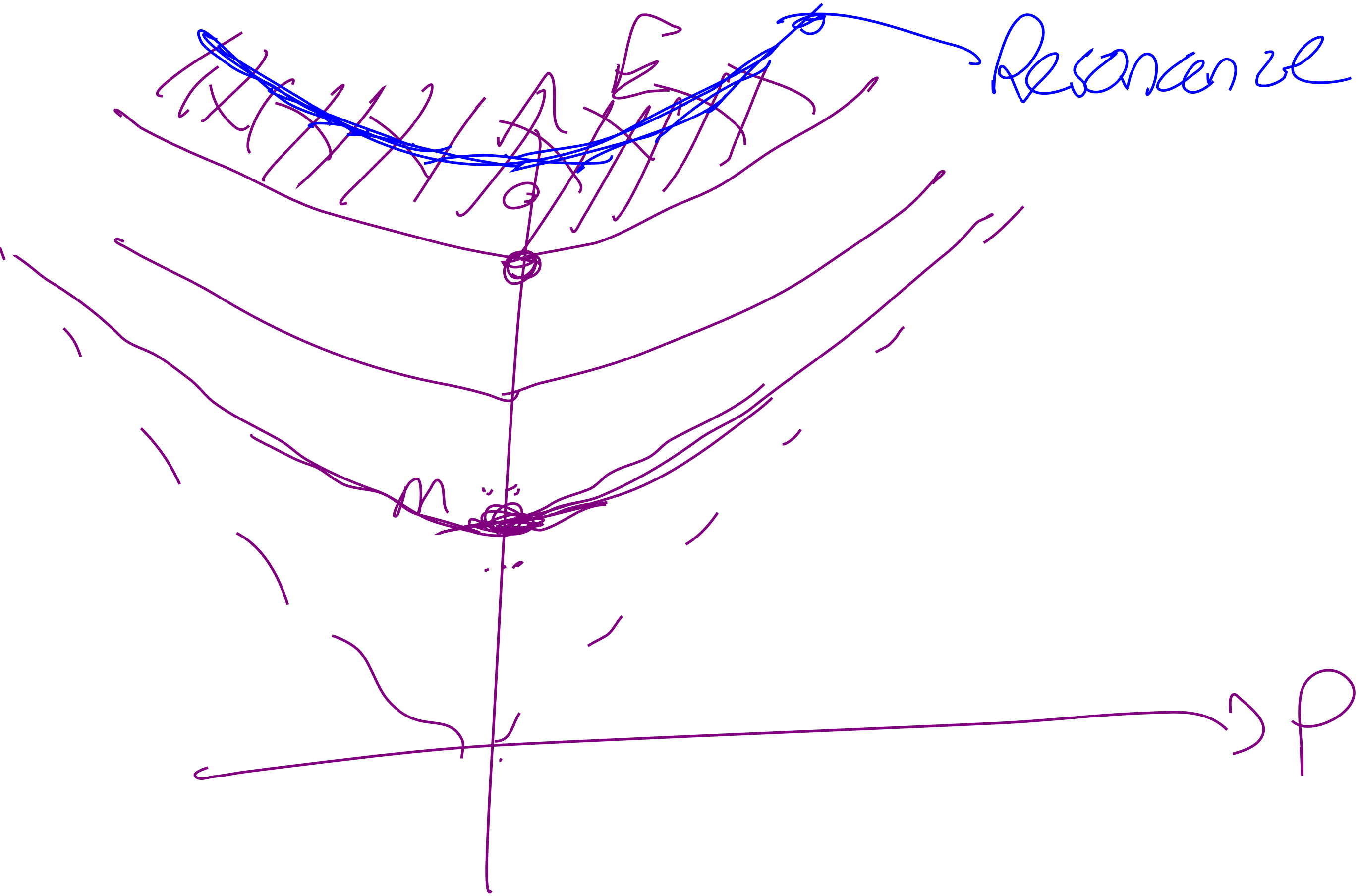
Free thy prop. from last time but with $m^2 \to s$

Correl. func. $\to$ sum of correl. of part's w. mass$^2 = s$, integr. over s with coeff. $\rho(s)$

Particle $m^2 = s_1$ or $m = \sqrt{s_1}$

Part. $m^2 = s_2$




E
Resonance
m
p

Correl. func's

$$\underline{G^>(y)} = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot y} G^>(p) \qquad \text{theory info}$$

$$G^>(p) = \int \frac{ds}{2\pi} \theta(p^0) 2\pi \delta(p^2 - s) \boxed{\rho(s)}$$

Lehmann Represent. of propagator

$$\rho(s) = \sum_n \delta(p_n^2 - s) \underbrace{c_n^* c_n}_{\text{positive semi def.}}$$

$\rho(s)$ is positive (semi) def.

~~Re~~ Stuff prop. as sum of prop. rules for part's. with positive weights.

Other interesting correl. funcs:

$$G^<(x) = G^>(-x) = \langle 0|\, \varphi(0)\varphi(x)\, |0\rangle$$

Answer = $G^>$ answer but with $\bar{\Theta}(-p^0)$

$G^>(x)$: $K_1(rm)$ spacelike $G^< = K_1(rm)$

$H_1(tm)$ timelike H_1^*

$$\sigma(x) = G^>(x) - G^<(x) = \langle 0|\, [\varphi(x), \varphi(0)]\, |0\rangle$$

Knows about causality:

$\sigma(x)$: $x^2 < 0 \Rightarrow 0$

spacesep

$G_R(x)$ Retarded Func. $\Rightarrow \sigma(x)\,\Theta(x^0)$

$\mathcal{L} \longrightarrow \mathcal{L} - \varphi(x)\, J(x)$ 　　 $J(x)$ is "external" you choose it func.

Modify in this way! 　　 xJ

Interaction picture $-\varphi J$ Int., regular $\mathcal{L}$ as $\mathcal{L}_0$

$$\boxed{H = H_0 + \varphi J}$$

What is $\langle 0 | e^{+iH(t-t_0)} \; \varphi(y)\, e^{-iH(t-t_0)} | 0 \rangle$ to linear order in J

distant past 　　 dist past

$$U(t,t_0) \simeq 1 - i \int_{-\infty}^{t} dt' \int d^3x\, J(x,t')\, \varphi(x,t')$$

$$U^\dagger \simeq 1 + \;\; " \;\; " \;\; " \;\; "$$

Linear order

$$\langle 0|\varphi(y)|0\rangle = \langle 0|\left[1 + i\int^{t'=t} d^4x\, J(x)\varphi(x)\right]\varphi(y)\left[1 - i\int^{t'=t} d^4x\, J(x)\varphi(x)\right]|0\rangle$$

$$+ \dots$$

$$\left(\langle\varphi\rangle = \langle 0|\varphi|0\rangle\right)$$

$$= \langle 0|\int d^4x\, J(x)\,\Theta(y^0 - x^0)\left(i\varphi(x)\varphi(y) - i\varphi(y)\varphi(x)\right)|0\rangle$$

$$= i\int d^4x\, J(x)\, G_R(y-x)$$

New G_R includes this i

$G^>$: are fields correlated?

G_R : if you "pull" on field how does it respond?

G : include causality

What's $G_R(p^\mu)$? $= \int d^4x \, e^{ip_\mu x^\mu} G_R(x)$

Note: as $x^0 > 0$, I am allowed to take p^0 to have Im

$\text{Im}\, p^0 > 0$ not just ≥ 0.

Find in Free Thy! $\langle 0|\varphi(y)\varphi(0)|0\rangle = \int dq \, e^{iq_\mu y^\mu} \Theta(q^0) \, 2\pi\delta(q^2 - m^2)$

$- \varphi(0)\varphi(y)$

$$G_R(p) = \int d^4y \, \Theta(y^0) e^{ip_\mu y^\mu} \int \frac{d^4q}{(2\pi)^4} \left[-i2\pi\delta(q^2 - m^2) \left(\Theta(q^0) - \Theta(-q^0) \right) e^{-iq_\mu y^\mu} \right]$$

$dt \, d^3y$ $\quad e^{ip^0y^0} e^{-i\vec{p}\cdot\vec{y}}$ $\quad \int dq^0 \, d^3q$ $\quad e^{-iq^0y^0} e^{+i\vec{q}\cdot\vec{y}}$

$(2\pi)^3 \delta^3(\vec{p} - \vec{q})$ $\quad \vec{q} \to \vec{p}$.

$$G_p(p) = \int dy^0\, \underline{\Theta}(y^0) \int \frac{dq^0}{\cancel{2\pi}} (-\cancel{2\pi} i)\, e^{i(p^0-q^0)y^0} \left(\underline{\Theta}(q^0) - \underline{\Theta}(-q^0) \right) \frac{\delta(q_0^2-\omega_p^2)}{}$$

$$\underline{\omega_p^2 = \vec{p}^{\,2} + m^2}$$

$$\int_0^\infty dy^0\, \underline{\Theta}(y^0) (-\cancel{2\pi} i) \left[e^{\boxed{i(p^0-\omega_p)}y^0} - e^{i(p^0+\omega_p)y^0} \right] \frac{1}{2\omega_p}$$

$$\boxed{\int_0^\infty dy^0\, e^{-Ay^0} = \frac{1}{A}}$$

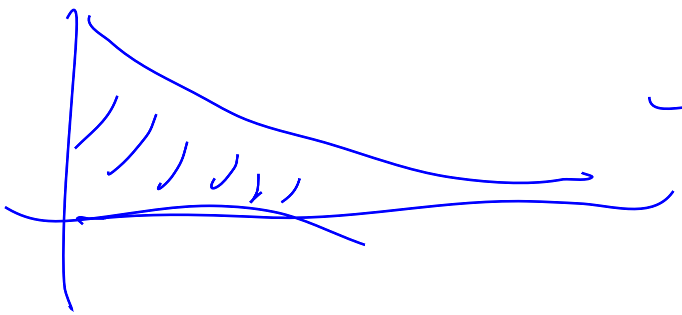
$$\frac{(-i)}{2\omega_p} \left[\frac{-1}{\cancel{i}(\omega_p - p^0)} + \frac{-1}{\cancel{i}(+\omega_p + p_0)} \right]$$

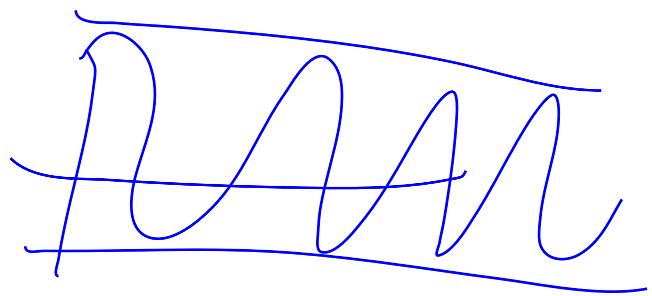
$$\frac{(-1)}{2\omega_p} \left(\frac{\omega_p + p_0 + \omega_p - p_0}{(\omega_p^2 - p_0^2)} \right) = \frac{2\omega_p}{2\omega_p(p_0^2 - \omega_p^2)} = \frac{1}{p^2 - m^2}$$

$$G_2(p^\mu) = \frac{1}{p^2 - m^2} = \frac{1}{(p_0)^2 - \vec{p}^{\,2} - m^2}$$

For p^0 with + im part.

$$\int e^{-At} \qquad \int e^{-At}\, \boxed{e^{ipt}} \qquad \int e^{ipt}\, dt$$





If p^0 has > im part ✓

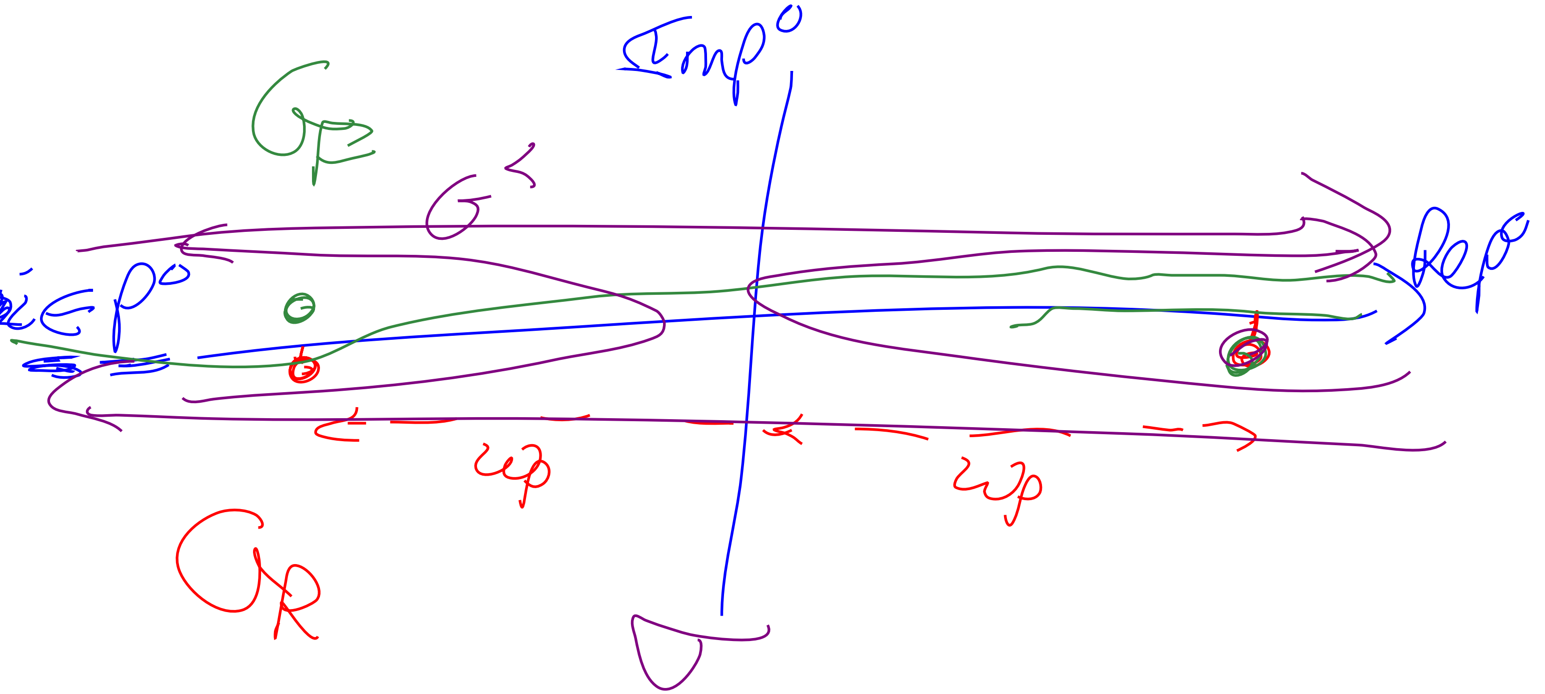
If not: $$\lim_{\epsilon \to 0} \frac{1}{(p_0 + i\epsilon)^2 - \vec{p}^{\,2} - m^2}$$

$$G_F(x) = -i\langle 0| \big(\varphi(x)\varphi(0)\Theta(x^0) \pm \varphi(0)\varphi(x)\Theta(-x^0)\big)|0\rangle$$

T-ordered correl. func.

$$G_R = \frac{1}{(p_0+i\epsilon)^2 - p^2 - m^2} \qquad G_F = \frac{1}{p_0^2 - p^2 - m^2 + i\epsilon}$$

$$= \frac{1}{p_0^2 - p^2 - m^2 + i\epsilon p^0}$$



$$G_R(p) = \int_{m^2}^{\infty} \frac{ds}{2\pi}\, \rho(s)\, \frac{1}{(p_0+i\epsilon)^2 - \vec{p}^{\,2} - s}$$