L&P 10 1

Problem 1: overall normalization.

I have no idea what $\tilde{N}^N$ is.

For most Purposes the overall norm. of the path $\int$ will not matter. We will see this in what follows.

(Basically $\langle 0|0\rangle = 1$ so I can always divide by $Z(J=0)$ and normalization cancels out. Whew!)

Problem 2: what is $\langle Q_0|0\rangle$ or $\langle 0|Q_N\rangle$?

I don't know. I know what the free theory vacuum is, but

Haag's Theorem: in either the $L \to \infty$ or $a \to 0$ limits,

$$\langle 0_{\text{free thy}} | 0_{\text{full thy}} \rangle \to 0.$$ Same with all $\langle \text{state}_{\text{full}} | \text{state}_{\text{free}} \rangle$

I don't know how to compute $\langle Q_N|0\rangle$ because I don't understand $|0\rangle$.

Solution: "slightly tilt contour."

I am doing $\langle 0| U(t_f - t_{mid1}) U(t_{mid1} - t_{mid2}) U(t_{mid2} - t_i)|0\rangle$

(Assume all interesting questions happen in some middle interval)

Make $\left\{ \begin{array}{l} t_f \text{ really big} \\ t_i \text{ really negative} \end{array} \right\}$ and then replace

$$e^{-iH(t_m - t_i)} \to e^{(-i-\varepsilon)H(t_m - t_i)} = e^{-iH(t_m-t_i)} e^{-\varepsilon H(t_m - t_i)}$$

choose ε such that $\left\{ \begin{array}{l} \varepsilon \ll 1 \text{ but} \\ \varepsilon(t_m - t_i) \gg 1/m \end{array} \right.$ (m = mass gap)

Why? Because

$$e^{-\beta H}|0\rangle = |0\rangle \qquad \text{with } \beta = \varepsilon(t_m - t_i)$$

but

$$e^{-\beta H}|n\rangle = e^{-E_n\beta}|n\rangle$$

if $E_n\beta \gg 1$ for <u>all</u> pos. energy states

then $e^{-E_n\beta} \simeq 0$.

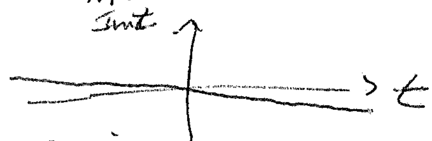
Put any old state - random mix of everything - as initial state:

$$|0\rangle \to |\psi\rangle = \underbrace{|0\rangle\langle 0|\psi\rangle}_{\text{as long as this not} = 0} + \sum_n |n\rangle\langle n|\psi\rangle$$

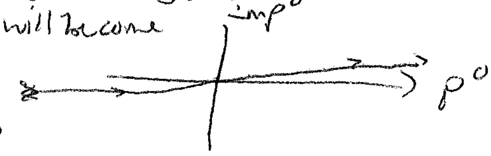
Applying $e^{-\beta H}$ <u>removes</u> all $|n\rangle$, leaves only $|0\rangle$

In fact, push $t_i \to -\infty$
$t_f \to +\infty$

with <u>infinitesimal</u> tilt $-i \to -i(1-i\varepsilon)$
in time contour



On Fourier transforming, that will become



Now forget about what init, final states are!
(or, trace over them!!)

But <u>remember</u>,

$$\int d^4x \, \mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$$

$t \to t(1-i\varepsilon)$

Can I do the path integral?

For the free theory, $\int \mathcal{D}\varphi \, \exp i \int d^4x \left(\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 \right)^{-J\varphi}$

Quadratic in φ

looks sorta like a Gaußian integral. Does that help?

Recall, one Gaußian integral

$$\int_{-\infty}^{\infty} dx \, e^{-x^2/2} = \sqrt{2\pi}$$

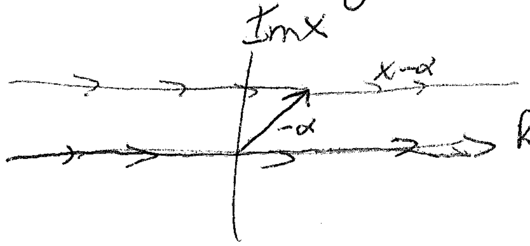
$$\int_{-\infty}^{\infty} dx \, e^{-\frac{x^2}{2} + \alpha x} = \int_{-\infty}^{\infty} dx \, e^{\overbrace{-\frac{x^2}{2} + \alpha x - \frac{\alpha^2}{2}}^{-\frac{1}{2}(x-\alpha)^2} + \frac{\alpha^2}{2}}$$

$$= e^{\alpha^2/2} \int_{-\infty}^{\infty} dx \, e^{-\frac{(x-\alpha)^2}{2}} \qquad y = x - \alpha . \int_{-\infty}^{\infty} e^{-y^2/2} = \sqrt{2\pi}$$

$$= \sqrt{2\pi} \, e^{\alpha^2/2}$$

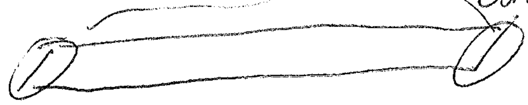
What if α is complex? Say, imaginary? Still true but must think about why.

$y = x - \alpha$ is a change of contour



Rely on 1) no singularities in the band $\mathrm{Im}\, x \in [0, -\mathrm{Im}\, \alpha]$

2) no contribution from ends of contour.



Change coordinates $y = M^{1/2} x$

$dy = M^{1/2} dx, \quad dx = M^{-1/2} dy$

$$\int dx_1 .. dx_N = \frac{1}{\sqrt{\text{Det} M}} \int dy_1 .. dy_N$$

Jacobian $1/\text{Det}\sqrt{M}$
$= 1/\sqrt{\text{Det}\, M}$

Completesquare:

$$\int dx_a \, e^{-\frac{1}{2} xMx - k \cdot x} = e^{+\frac{1}{2} k M^{-1} k} \frac{(2\pi)^{N/2}}{\text{Det}^{1/2} M}$$

so long as all eigenvalues of M have $\text{Re}\,\lambda > 0$

(Note in particular, $\int e^{(i-\varepsilon)\frac{x^2}{2}} dx = \frac{\sqrt{2\pi}}{\sqrt{-i+\varepsilon}} = \left(\frac{1+i}{\sqrt{2}}\right)\sqrt{2\pi}$ ...

<u>What about</u> $$\int dx_1 .. dx_N \; x_{a_1} x_{a_2} .. x_{a_m} \, e^{-\frac{1}{2} xMx} \quad ?$$

well, $$\frac{\partial}{\partial J_b} e^{x_a J_a} = x_b e^{x_a J_a}$$

$$\left[\frac{\partial^m}{\partial J_{a_1} .. \partial J_{a_m}} e^{\sum_a x_a J_a}\right]\Bigg|_{J=0} = \left[x_{a_1} x_{a_2} .. x_{a_m} e^{\sum x_a J_a}\right]\Bigg|_{J=0}$$

$$= x_{a_1} .. x_{a_m}.$$

What I want $$= \left[\frac{\partial^m}{\partial J_{a_1} .. \partial J_{a_m}} \int dx_1 .. dx_N \, e^{-\frac{1}{2} xMx} e^{x_a J_a}\right]\Bigg|_{J=0}$$

$$= \left[\frac{\partial^m}{\partial J_{a_1} .. \partial J_{a_m}} \frac{(2\pi)^{N/2}}{\text{Det}^{1/2} M} e^{\frac{1}{2} J_a M^{-1}_{ab} J_b}\right]\Bigg|_{J=0}$$

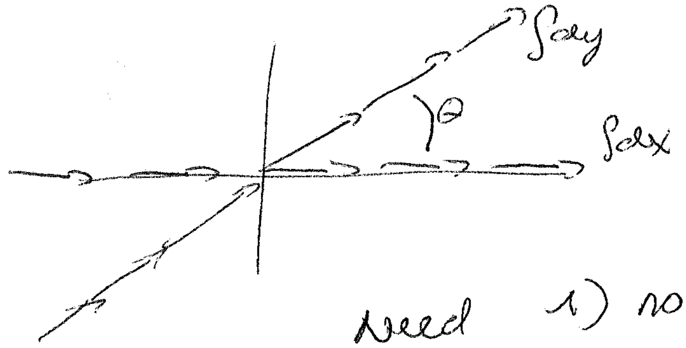
What about $\int_{-\infty}^{\infty} dx\, e^{-\frac{mx^3}{2}}$? $y = \sqrt{m}\, x$

$dy = \sqrt{m}\, x$, $dx = \frac{1}{\sqrt{m}} dy$

$$\frac{1}{\sqrt{m}} \int_{-\infty}^{\infty} dy\, e^{-y^2/2} = \frac{\sqrt{2\pi}}{\sqrt{m}}$$

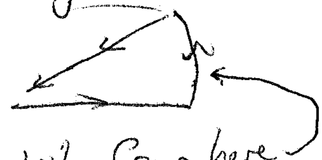
$\frac{1}{\sqrt{m}}$ = Jacobian of basis change

What if m is complex!? $y = \sqrt{m}\, x$ is a contour change



$\Theta = \frac{1}{2} \operatorname{Arg} m$

Need 1) no singularities



2 no contrib. from here

No contrib. from arc provided that $Re(m) > 0$ however small.

Actually, $Re(m) \geq 0$ also barely OK.

My $t \to t(1 - i\varepsilon)$ will provide this for path integral...

What about many variables,

$$\int \prod_{a=1}^{N} dx_a \exp\left(-\sum_{ab=1}^{N} \frac{1}{2} M_{ab} x_a x_b + \sum_{a=1}^{N} K_a x_a\right)$$ General Quadratic Form

Note: M_{ab} symmetric ~~(if real) or at least~~

Provided eigenvalues of M_{ab} all have Re parts ≥ 0

M^{-1} is defined as is $M^{+1/2}$: use root with + Re part

Define $y_a = M_{ab}^{+1/2} x_b$

$x_a M_{ab} x_b = y_a y_a$ and $K_a x_a = K_a M_{ab}^{-1/2} y_b$

What I typically want is

$$\langle 0 | \varphi_{a_1} \cdots \varphi_{a_m} | 0 \rangle = \frac{\int \varphi_{a_1} \cdots \varphi_{a_m} e^{\sim}}{\int e^{\sim}}$$

denominator to cancel normalization factor.

$$\frac{1}{Z(0)} \frac{\partial^m}{\partial J_{a_1} \cdots \partial J_{a_m}} Z(J) \Big|_{J=0}$$

and $\frac{Z(J)}{Z(0)}$ is just the $e^{\frac{1}{2} J_a M_{ab} J_b}$ piece.

$$\text{Correlation function} = \frac{\partial^m}{\partial J_{a_1} \cdots \partial J_{a_m}} e^{\frac{1}{2} J M^{-1} J}$$

Wick's Theorem: this $= \sum_{\substack{\text{all pairings} \\ \text{of the } a_i}} M^{-1}_{a_{p_1} a_{p_2}} M^{-1}_{a_{p_3} a_{p_4}} \cdots M^{-1}_{a_{p_{m-1}} a_{p_m}}$

2 things: $M^{-1}_{a_1 a_2}$ 1

4 things: $M^{-1}_{a_1 a_2} M^{-1}_{a_3 a_4} + M^{-1}_{a_1 a_3} M^{-1}_{a_2 a_4} + M^{-1}_{a_1 a_4} M^{-1}_{a_2 a_3}$ $1 \cdot 3$

6 things: $M^{-1}_{a_1 a_2} M^{-1}_{a_3 a_4} M^{-1}_{a_5 a_6} + \ldots$ 15 total comb $1 \cdot 3 \cdot 5$

$1 \cdot 3 \cdot 5 \cdot 7$

2n objects: $(2n-1)!! \equiv \frac{(2n)!}{2^n n!}$ pairings

Proof: induction.

Outline of proof

I want $\left(\frac{\partial^n}{\partial J_{a_1} \cdots \partial J_{a_n}} e^{+\frac{1}{2} J_a M^{-1}_{ab} J_b} \right) \Big|_{J=0}$

Note that $\left(J_{a_i} \frac{\partial^m}{\partial \cdots} e^{\cdots} \right) \Big|_{J=0} = 0$ because the J_{a_i} will not go away.

Plan: apply $\frac{\partial}{\partial J_{a_n}}$ to the exponential: $\frac{\partial}{\partial J_{a_n}} e^{\frac{1}{2}()} = M^{-1}_{a_n b} J_b \, e^{\frac{1}{2}(\dots)}$

Now move the J_b past all other derivatives:

$$\frac{\delta}{\delta J_{a_m}} J_b = \delta_{a_m, b} + J_b \frac{\delta}{\delta J_{a_m}}$$

Generates $(n-1)$ terms

$$\sum_{i=1}^{n-1} M^{-1}_{a_n a_i} \frac{\delta^{n-2}}{\underbrace{\delta J_{a_1} \cdots \delta J_{a_{n-1}}}_{\text{with } J_{a_i} \text{ missing}}} e^{\frac{1}{2} J M^{-1} J} \Bigg|_{J=0}$$

all possibilities how a_n can pair.

Apply iteratively ....