$\angle$ 11 P1

Now consider $Z(J) = \int \mathcal{D}\varphi \exp i \int d^4x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - J(x)\varphi(x) \right]$

Think for instance for $x_i = a n_i$

$$\partial_i \varphi = \frac{\varphi(x+a\hat{i}) - \varphi(x)}{a} \qquad \partial_i\varphi\partial^i\varphi = -\frac{(\varphi(x+a\hat{i}) - \varphi(x))^2}{a^2}$$

$$\partial_\mu\varphi\partial^\mu\varphi = \sum_{\mu=0}^{3} (-1)^{\text{if } \mu\neq 0} \left(\frac{1}{2}\varphi^2(x+a\hat{\mu}) + \frac{1}{2}\varphi^2(x) - \varphi(x+a\hat{\mu})\varphi(x) \right)$$

$\int d^4x \;=\; \sum_x$ ~~$= \sum \frac{1}{2}\varphi(x)\{8\varphi(x) - \sum_\mu \varphi(x$~~ ~~rearranging~~

$$= \frac{1}{a^2}\sum_x \left\{ \begin{array}{l} \frac{1}{2}\varphi(x)\{2\varphi(x) - \varphi(x+a\hat{t}) - \varphi(x-a\hat{t})\} \\ \left(-\sum_i \frac{1}{2}\varphi(x)\{2\varphi(x) - \varphi(x+a\hat{i}) - \varphi(x-a\hat{i})\} \right) \end{array} \right\}$$

in terms of $x, y \equiv x$ or $x \pm a\hat{\mu}$, this is like

$[\varphi(x-3a\hat{\mu})\ \varphi(x-2a\hat{\mu}) \ldots]$

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} \varphi(x-3\hat{a}) \\ \varphi(x-2\hat{a}) \\ \varphi(x-\hat{a}) \\ \varphi(x) \\ \varphi(x+\hat{a}) \\ \vdots \end{bmatrix}$$

I can rewrite it as

$$\sum_{x,y} \varphi(x) \left[\sum^{\mu} \frac{1}{2a^2}\left(2\delta_{xy} - \delta_{x,y+a\hat{\mu}} - \delta_{x,y-a\hat{\mu}}\right) \right] \varphi(y)$$

$\hookrightarrow$ Discrete version of $-\partial_\mu\partial^\mu$

The point is that I can think of $-\frac{\partial_\mu \partial^\mu}{2}$ as a ~~(nonlocal)~~ (not-quite-local) (differential) operator

$$\int d^4x \, -\tfrac{1}{2}\varphi \partial_\mu \partial^\mu \varphi = \int d^4x\, d^4y \; \varphi(x) M(x,y) \varphi(y)$$

$$M(x,y) = -\tfrac{1}{2}\partial_\mu \partial^\mu \delta^4(x-y) \quad \text{formally.}$$

So $$iS = \frac{i}{2}\underbrace{\int d^4x\, d^4y}_{\sum_{ab}} \varphi(x) \underbrace{\left[\left(-\tfrac{1}{2}\partial_\mu \partial^\mu - m^2\right)\delta^4(x-y)\right]}_{\text{“}M\text{”}} \varphi(y)$$

of form $\sum_{ab} \frac{1}{2} X_a M_{ab} X_b$. Similarly $i\int -\varphi(x)J(x) \sim \sum_a X_a K_a$ form.

So,

$$Z_{free}(J) = \int \mathcal{D}\varphi \, e^{i\int_{xy}\frac{1}{2}\varphi(-\partial^2 - m^2)\varphi} \, e^{-i\int J\varphi}$$

$$= \exp\left[\frac{i}{2}\int_{xy} J(x) \left[\partial^2 + m^2\right]^{-1}_{xy} J(y)\right]$$

What is $(\partial^2 + m^2)^{-1}$??

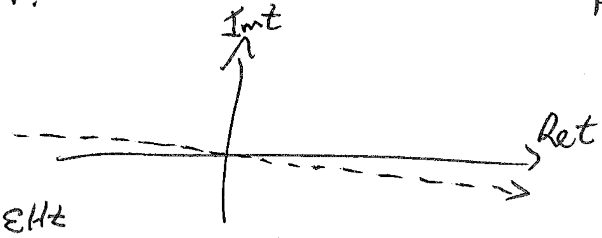
If ∂^2 is like $\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & \\ & & & \ddots \end{bmatrix}$, $(\partial^2)^{-1}$ is matrix which hits this and gives $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$

That is, $(\partial^2 + m^2)^{-1}_{xy} (\partial^2 + m^2)_{yz} = \delta^4(x-z)$

Claim: this is $-\Delta_F$, e.g., in momentum space it's $\frac{-1}{p^2 - m^2 + i\epsilon}$

Really? And why this $i\varepsilon$??

Recall that time follows



so that $e^{-iHt} \to e^{-iHt - \varepsilon H t}$ to remove non-vacuum states.

That means $\int dt \to \int (1-i\varepsilon)\,dt$

$$\frac{d}{dt} \to \frac{d}{(1-i\varepsilon)\,dt}$$

Another way to find $\Delta(x-y) = (\partial^2 + m^2)^{-1}_{x,y}$ is to use

$$\frac{i\delta}{\delta J(x)}\frac{i\delta}{\delta J(y)} Z(J) = \int \mathcal{D}\varphi\ \varphi(x)\varphi(y)\, e^{i(S + \int J\varphi)}$$

$$= i(\partial^2 + m^2)^{-1}_{xy}.$$

Let's do the Gaußian integral ~~with $\varphi(x)\varphi(y)$~~.

Only – instead of working in coord, we may as well Fourier transform to $\varphi(p)$, $J(p)$!!

$$+i\Delta(p) = \int d^4x\, e^{ip\cdot x} \int \mathcal{D}\varphi\ \frac{-\delta^2}{\delta J(x)\,\delta J(0)} \cancel{\varphi(x)\varphi(0)}\ e^{i\int d^3y\, dt \left[\begin{array}{l}(1-i\varepsilon)\varphi(+\nabla^2 - m^2)\varphi \\ +(1/_{1-i\varepsilon})\,\varphi(-\partial_t^2)\varphi\end{array}\right]}\, e^{-i\int J\varphi}$$

$\int \mathcal{D}\varphi$ is same in x- or p-basis

for $i\int d^3y\, dt\left[(1-i\varepsilon)\,\varphi(y,t)(\nabla^2 - m^2)\varphi(y,t) + \frac{-1}{1-i\varepsilon}\varphi(y,t)\partial_t^2\varphi(y,t)\right]$

$$\varphi(y,t) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot y}\,\tilde{\varphi}(q)$$

get $iS = i\int \frac{d^4q}{(2\pi)^4}\left[(1-i\varepsilon)(-m^2 - \vec{q}^{\,2})\,\tilde{\varphi}(q)\tilde{\varphi}(-q) + \frac{q_0^2}{1-i\varepsilon}\,\tilde{\varphi}(q)\tilde{\varphi}(-q)\right]$

Leads to

$$\frac{i}{(1-i\varepsilon)(-\vec{q}^{\,2}-m^2)+\frac{1}{1-i\varepsilon}(q^0)^2} = \frac{i}{q^{0\,2}-\vec{q}^{\,2}-m^2+i\varepsilon(q^{0\,2}+\vec{q}^{\,2}+m^2)}$$

positive

The $i\varepsilon$ prescription arises from the contour rotation.

Fine. But I want

$$Z(J) = \int \mathcal{D}\varphi \exp i\int_x \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - J\varphi - \frac{\lambda}{4!}\varphi^4$$

or more generally $e^{iS_{\text{Free}}}e^{iS_I}$

Now what?

Perturbation Theory Method

1) Assume λ small so $\frac{\lambda}{4!}\varphi^4$ term is "perturbation" and I can try to expand order by order in λ

2) write

$$Z(J) = \int \mathcal{D}\varphi\, e^{i\int_x -\frac{\lambda}{4!}\varphi^4(x)}\, e^{i\int d^4y\left(\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - m^2\varphi^2\right)}\, e^{-i\int_z J(z)\varphi(z)}$$

3) Note that $\varphi(x)e^{-i\int_y J(y)\varphi(y)} = \frac{i\delta}{\delta J(x)} e^{-i\int_y J(y)\varphi(y)}$ (check!!)

$$F(\varphi)e^{-i\int_y J(y)\varphi(y)} = F\left(\frac{i\delta}{\delta J}\right)e^{-i\int_y J(y)\varphi(y)}$$

4) $e^{i\int_x -\frac{\lambda}{4!}\left(\frac{i\delta}{\delta J(x)}\right)^4}$ is φ-independent. Pull it out of the integral!

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Now I can do the remaining Gaußian path $\int$:

$$Z(J) = \int \mathcal{D}\varphi\, e^{\frac{-i\lambda}{4!}\int_x \varphi^4}\, e^{i\int (\partial_\mu\varphi \partial^\mu\varphi - m^2\varphi^2) - J\varphi}$$

$$= e^{\frac{-i\lambda}{4!}\int_x \left(\frac{i\delta}{\delta J(x)}\right)^4} \int \mathcal{D}\varphi\, e^{i\int \varphi\left(-\frac{\Delta^{-1}}{2}\right)\varphi}\, e^{-i\int J\varphi}$$

or $e^{iS_I\left(\frac{i\delta}{\delta J}\right)}$ in general

$$= e^{\frac{-i\lambda}{4!}\int_x \left(\frac{i\delta}{\delta J(x)}\right)^4}\, \cancel{e^{\frac{i}{2}J\Delta J}}\, e^{\frac{i}{2}\int d^4y\, d^4z\, J(y)\Delta_F(y-z)J(z)}$$

5) Replace with its series expansion in λ

$$Z(J) = \sum_{n=0}^{\infty} \left(\frac{-i\lambda}{4!}\right)^n \int d^4x_1 \ldots d^4x_n \left(\frac{i^{4n}\delta^{4n}}{(\delta J(x_1))^4 \ldots (\delta J(x_n))^4}\right) e^{\frac{i}{2}\int_{yz} J(y)\Delta_F(y-z)J(z)}$$

Can you really do this?

Two problematic steps:
- Expand function e^{iS_I} in a series
- Reverse order of integration and series expansion

Math types tell us: be very careful - only well defined & valid if some convergence properties hold.

Do they? NO! But see the homework!!

Series is useful anyways!!