Doing Something with Perturbative Expansion

Q: what is (differential) cross-section for 2 scalars, momenta P_1, P_2, to scatter to momenta K_1, K_2 ?
($p^2 = m^2 = k^2$ in each case)

A: $$\underbrace{\int \frac{d^3K_1\, d^3K_2}{(2\pi)^3\, 2K_1^0\, 2K_2^0}}_{\text{range....}} \frac{1}{2P_1^0\, 2P_2^0\, |v_1 - v_2|} |\mathcal{M}|^2 (2\pi)^4 \delta^4(P_1 + P_2 - K_1 - K_2)$$

with $$\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{i(P_1\cdot x_1 + P_2\cdot x_2 - K_1\cdot x_3 - K_2\cdot x_4)} \quad \cancel{\langle 0| T(x_4, x_3, x_2,}$$

$$\times \langle 0| T\big(\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\big)|0\rangle$$

$$= \frac{i^4}{(P_1^2 - m^2 + i\varepsilon)(P_2^2 - m^2 + i\varepsilon)(K_1^2 - m^2 + i\varepsilon)(K_2^2 - m^2 + i\varepsilon)} (2\pi)^4 \delta^4(P_1 + P_2 - K_1 - K_2)\, i\mathcal{M}(P_1 P_2; K_1 K_2)$$

So let's find

$$\langle 0| T\big(\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\big)|0\rangle$$

$$= \frac{i^4\, \delta^4}{\delta J(x_1) \cdots \delta J(x_4)} \sum_{n=0}^{\infty} \left(\frac{-i\lambda}{24}\right)^n \frac{1}{n!} \int d^4y_1 \cdots d^4y_n \left(\frac{i\delta}{\delta J(y_1)}\right)^4 \cdots \left(\frac{i\delta}{\delta J(y_n)}\right)^4$$

$$\exp \frac{-i}{2} \int d^4z_1\, d^4z_2\, J(z_1)\, \Delta_F(z_1 - z_2)\, J(z_2)$$

Lowest order

$$\frac{i^4 \delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} e^{-\frac{i}{2}\int J\Delta J} = i^6 \big(\Delta(x_1-x_2)\Delta(x_3-x_4) + \Delta(x_1-x_3)\Delta(x_2-x_4) + \Delta(x_1-x_4)\Delta(x_2-x_3)\big)$$

Think about $\int d^4x_1 d^4x_3 \, e^{i(p_1\cdot x_1 - k_1\cdot x_3)} \Delta(x_1-x_3)$

introduce $r = x_1 - x_3$ relative, cm coord

$c = \frac{x_1+x_3}{2}$ $\qquad d^4x_1 d^4x_3 = d^4r\, d^4c$

$x_1 = c + r/2$

$x_3 = c - r/2$

$$\int d^4c \, e^{ic\cdot(p_1-k_1)} \int d^4r \, e^{ir\left(\frac{p_1+k_1}{2}\right)} \Delta(r)$$

$$= (2\pi)^4 \delta^4(p_1-k_1)\, \Delta\left(\frac{p_1+k_1}{2}\right) = (2\pi)^4 \delta^4(p_1-k_1)\,\Delta(p_1)$$

forces $p_1 = k_1$ and gives a propagator. $\Delta(x_2-x_4) \to \delta(p_2-k_2)$ also.

But if $p_1 = k_1$, $p_2 = k_2$ then no scattering happened!!

These terms keep track of free propagation for $p = k$

Probability that nothing happens. Silly.

Next order

$$\frac{i^4 \delta^4}{\delta J(x_1)\cdots\delta J(x_4)} \; \frac{-i\lambda}{24}\int d^4y \, \frac{i^4\delta^4}{(\delta J(y))^4} \, e^{-\frac{i}{2}\int_{zw} J(z)\Delta(z-w)J(w)}$$

All pairings of $8 \; \frac{\delta}{\delta J}$'s : $7\cdot 5\cdot 3\cdot 1 = 105$ terms. Eek!

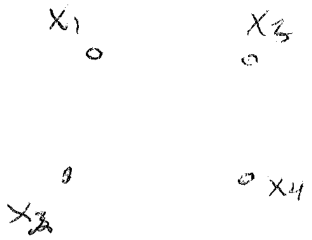
Draw picture.

Each $\frac{i\delta}{\delta J(x)}$ is a dot $\overset{x}{\circ}$ $\qquad \frac{i\delta}{\delta J}\frac{i\delta}{\delta J} e^{-\frac{iJ\Delta J}{2}} = +i\Delta(x-y)$

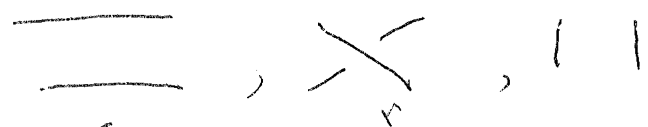
connects points x and y

the $\frac{i\delta^4}{(\delta J(y))^4}$ is a $\underset{y}{\times}$ $\qquad$ So draw a line from one to other.

$$\frac{i^4\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}$$

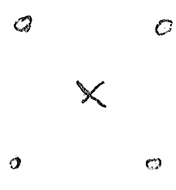


connect all dots. 3 ways

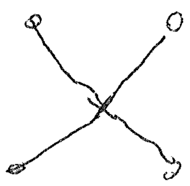


$i^2\Delta(x_1-x_3)\Delta(x_2-x_4)$ $\quad i^2\Delta(x_1-x_4)\Delta(x_2-x_3)$ $\quad i^2\Delta(x_1-x_2)\Delta(x_3-x_4)$

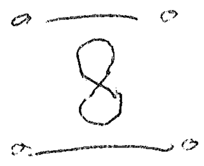
$$\frac{i^4\delta^4}{\delta J(x_1)\cdots\delta J(x_4)} \left(\frac{-i\lambda}{24}\right)\int d^4y \left(\frac{i\delta}{\delta J(y)}\right)^4$$



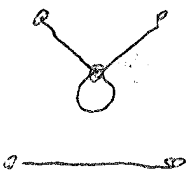
7 distinct ways



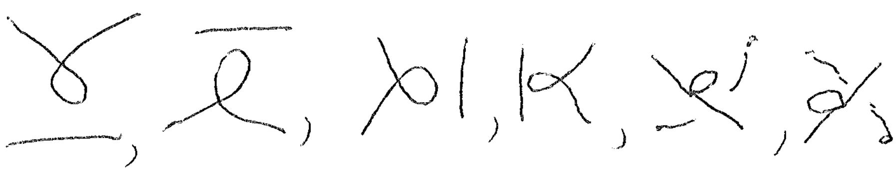
(24 versions)



3 versions each : × 3 choices

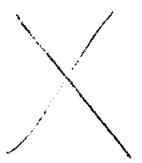


12 versions, 6 ways



$24 + 9 + 72 = 105$ good! we got them all!

Now, what do they mean?



$$-\frac{i\lambda}{24}\int d^4y \; \Delta_F(x_1-y)\Delta_F(x_2-y)\Delta_F(x_3-y)\Delta_F(x_4-y)\; i^4 \quad \times 24$$

\# of ways of connecting the $\frac{\delta}{\delta J(x)}$'s with $\frac{\delta}{\delta J(y)}$'s

Let's Fourier transform this!

$$(-i\lambda)\int d^4x_1 d^4x_2 d^4x_3 d^2x_4 \, e^{i(p_1\cdot x_1 + p_2\cdot x_2 - k_1\cdot x_3 - k_2\cdot x_4)} \int d^4y \; \Delta_F(x_1-y)\Delta_F(x_2-y) \Delta_F(x_3-y)\Delta_F(x_4-y)$$

write $\Delta_F(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \Delta_F(q)$

Aha! now $\int d^4x_1 \, e^{ip_1\cdot x_1} e^{-iq\cdot x_1} = (2\pi)^4\delta^4(p_1-q)$. Will perform all q-int's

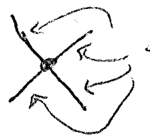
$$(-i\lambda)\, \Delta_F(p_1)\Delta_F(p_2)\Delta_F(-k_1)\Delta_F(-k_2) \int d^4y \, e^{iy\cdot(p_1+p_2-k_1-k_2)}$$

$$= \Delta_F(p_1)\Delta_F(p_2)\Delta_F(-k_1)\Delta_F(-k_2)\,(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\,(-i\lambda)$$

Recall, $\langle \varphi\varphi\varphi\varphi \rangle = \Delta_F(P_1)\Delta_F(P_2)\Delta_F(P_3)\Delta_F(P_4)(2\pi)^4\delta^4(P_1+P_2-K_1-K_2)$
$\times -i\mathcal{M}(P_1,P_2;K_1,K_2)$

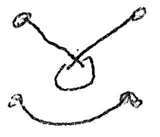
So at this order, $\mathcal{M}=\lambda$. Aha!

And look, my $\Delta(p)$'s, $(2\pi)^4\delta^4(p\text{'s}-k\text{'s})$ actually showed up!



these always appear & always give the $\Delta(p)$ singular factors I want.

What of all that other stuff?



$$-\frac{i\lambda}{2}\int d^4y\, \Delta_F(x_1-y)\Delta_F(x_3-y)\Delta_F(x_2-x_4)\Delta_F(y-y)$$

Fourier transform: $\left(-\frac{i\lambda}{2}\right)\Delta_F(P_1)\Delta_F(K_1)(2\pi)^4\delta^4(P_1-K_1)$
$(2\pi)^4\delta^4(P_2-K_2)\,\Delta_F(P_2)$
$\times\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)$

looks like = *an extra factor of $\left(-\frac{i\lambda}{2}\right)\Delta_F(P_1)\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)$

huh?

Well, if I compute the full 2-pt function to $\mathcal{O}(\lambda)$,

$$\frac{i^2\delta^2}{\delta J(x_1)\delta J(x_2)}\left(\sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{n!(4!)^n}\prod_i\int d^4y_i\left(\frac{i\delta}{\delta J(y_i)}\right)^4\right)e^{-\frac{iJ\Delta J}{2}}$$

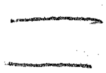
lowest order $\rightarrow$ $i\Delta(x_1-y_1)$

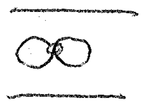
next order $\rightarrow$ $-\frac{i\lambda}{2}\int d^4y\, i\Delta_F(x_1-y)\,i\Delta_F(x_2-y)\,i\Delta_F(y-y)$

aha - so this stuff is "just" the $\mathcal{O}(\lambda)$ correction to the 2-pt function.

$\langle \varphi\varphi \rangle \equiv iG(x-y) = i\Delta(x-y) -$ corrections

this is the first such correction.

It means  has each $\Delta \to \Delta +$ first correction!

But what about  $-\frac{i\lambda}{8}\int d^4y\, \Delta(y-y)\Delta(y-y)\Delta(x_1-x_3)\Delta(x_2-x_4)$

well $-\frac{i\lambda}{8}\int d^4y\, \Delta(y-y)\Delta(y-y) = \infty$ is just some overall constant

Note: it is also extensive: $-\frac{i\lambda}{8}\Delta^2(0)\int d^4y \approx Vt = \Omega$

vol of space-time.

huh??

Oh I forgot! I want $\frac{1}{Z(J=0)}\frac{i^4\delta^4}{(\delta J(x_1)\cdots}\cdots\cdots$

$\boxed{\text{what is } Z(J=0)\,??}$