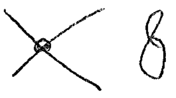
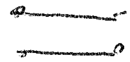
Applying what we learned last time

1) There were these stupid processes 

Vacuum bubbles - ~~too~~ factors of $\Omega = Vt$

Represent vacuum energy, canceled by $Z^{-1}(0)$ factor

2) There were boring processes 

too many Energy-Momentum δ functions.

Represent some particles not scattering

Fix: Define $W(J) = \ln Z(J)$

claim: $\dfrac{i^n \delta^n}{\delta J_{a_1}(x_1) \dots \delta J_{a_n}(x_n)} W(J)$ = all <u>connected</u> diagrams, with these $=$-types absent.

For instance at the free level $W = \frac{-i}{2} \int_{z_1 z_2} J(z_1)\Delta(z_1,z_2)J(z_2)$

clearly more than 2 variations vanish.

In detail: ~~for a the~~ define $G^{(n)}_{(x_1,\dots,x_n)} = \langle 0|\varphi(x_1)\dots\varphi(x_n)|0\rangle$

$G_c^{(1)}(x_1) = G^{(1)}(x_1)$

$$G_c^{(2)}(x_1,x_2) = G^{(2)}(x_1,x_2) - G_c^{(1)}(x_1)G_c^{(1)}(x_2)$$

$$G_c^{(3)}(x_1,x_2,x_3) = G^{(3)}(x_1,x_2,x_3) - G_c^{(1)}(x_1)G_c^{(1)}(x_2)G_c^{(1)}(x_3)$$

$$- G_c^{(1)}(x)G_c^{(2)}(x_1,x_2) - 2 \text{ other comb's}$$

$$G_c^{(4)}(x_1 \dots x_4) = G^{(4)} - (G^{(1)})^4 - (G_c^{1})^2 G_c^{(2)} \text{ (3 versions)}$$

$$- (G^{(2)})^2 \text{ (3 versions)} - G_c^{(1)} G_c^{(3)} \text{ (4 versions)} \dots$$

Lessom 8: Diagrams will give rise, on external lines, of the full (incoming) propagators, not just the free ones. That's ok.

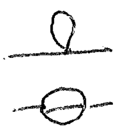
Summary of last time

1) Draw one ~~and~~ external – for each $\frac{i\delta}{\delta J}$ (eg. each $\varphi(x)$)

2) Draw 0, 1, 2, ... $\times$ representing the # of $\lambda\varphi^4$ insertions

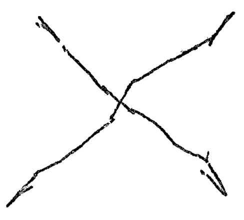
3) Find all ways to connect them.

Ignore ∞ ~~...~~ vacuum bubbles and don't worry about



disconnected graphs.

4) Convert a diagram into a value:



each external has a location $X_1 .. X_4$

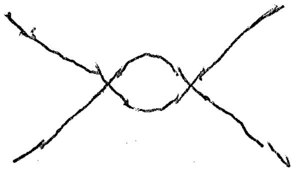
each internal has varying location $\int d^4y$

Each line is $i\Delta(x-y)$ between endpoints

overall factor $\left(\frac{-i\lambda}{24}\right)$ pervertex, $\frac{1}{n!}$ for # vertices

$\times$ number of ways of assembling pieces $= (-i\lambda)^n \cdot \frac{1}{\text{symm-factor}}$

Example: at $\mathcal{O}(\lambda^2)$, one process is



ways to assemble: $8 \times 3 \times 4 \times 3 \times 2 = 24 \cdot 24$

coefficient: $\frac{1}{24} \cdot \frac{1}{24} \cdot \frac{1}{2} \times 24 \times 24 = \frac{1}{2}$ ✓

Value: $\left(\frac{1}{2}\right)\left(-i\lambda\right)^2 \int d^4y_1 d^4y_2 \; i\Delta(x_1-y_1)\, i\Delta(x_2-y_1)$
$i\Delta(y_1-y_2)\, i\Delta(y_1-y_2)$
$i\Delta(y_2-x_3)\, i\Delta(y_2-x_4)$

Fourier transform:

each external line's x gets $\int d^4x \, e^{ip\cdot x}$

Each $\Delta(x_1-x_2)$ can be written $\int \frac{d^4q}{(2\pi)^4} \Delta(q) e^{-iq\cdot x_1} e^{+iq\cdot x_2}$

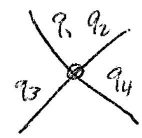
Each $\Delta \rightarrow$ a momentum integral

Each external $\int d^4x \, e^{ip\cdot x}$ ($\times e^{-iq\cdot x}$ from propagator)

is $(2\pi)^4 \delta^4(\text{propagator } q \; \underbrace{-\, p_{\text{external}}})$ $\begin{cases} - \text{ for incoming} \\ + \text{ for outgoing} \end{cases}$

Each $\int d^4y \rightarrow (2\pi)^4 \delta^4(\Sigma \text{ momenta entering vertex})$

example 1: from last time



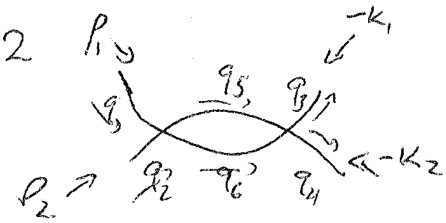
$$(-i\lambda)(i^4)\underbrace{\int \frac{d^4q_1\, d^4q_2\, d^4q_3\, d^4q_4}{(2\pi)^{4\times 4}}\Delta(q_1)\ldots\Delta(q_4)}_{\text{propagators}}\times \underbrace{\begin{matrix}(2\pi)^4\delta^4(p_1-q_1)\\(2\pi)^4\delta^4(p_2-q_2)\\(2\pi)^4\delta^4(-k_1-q_3)\\(2\pi)^4\delta^4(-k_2-q_4)\end{matrix}}_{\text{externals}}$$

$$\times \underbrace{(2\pi)^4\delta^4(q_1+q_2+q_3+q_4)}_{y\text{-integral (vertex)}}$$

$$= (-i\lambda)\, i\Delta(p_1)\, i\Delta(p_2)\, i\Delta(-k_1)\, i\Delta(-k_2)\,(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)$$

4 propagators: 4 integrals. 4 externals - 4 δ's 1 vertex - 1 more δ.

example 2



note: choose direction for each q! but choice is arbitrary

$$\left(\tfrac{1}{2}\right)(-i\lambda)^2\int\frac{d^4q_1\ldots d^4q_6}{(2\pi)^{4\cdot 6}}\, i\Delta(q_1)\ldots i\Delta(q_6)\times \begin{matrix}(2\pi)^4\delta^4(q_1-p_1)\\ \vdots \\ (2\pi)^4\delta^4(q_4+k_4)\end{matrix}\times\begin{matrix}(2\pi)^4\delta^4(q_1+q_2-q_5-q_6)\\ (2\pi)^4\delta^4(q_5+q_6-q_3-q_4)\end{matrix}$$

δ's: $q_1=p_1$ $\quad p_1+p_2=q_5+q_6$
$q_2=p_2$ $\quad k_1+k_2=-q_5-q_6$
$q_3=-k_1$
$\underbrace{q_4=-k_2}_{\text{do 4 of 6 int's}}$

can re-arrange δ's: replace (q_5+q_6) with (p_1+p_2):

$$(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\,\underbrace{(2\pi)^4\delta^4(q_1+q_2-q_5-q_6)}_{\text{does one more int}}$$

one int. left undone!

$$\left(\tfrac{1}{2}\right)(-i\lambda)^2\, i\Delta(p_1)\ldots i\Delta(k_2)\int\frac{d^4q_5}{(2\pi)^4}\, i\Delta(q_5)\, i\Delta(p_1+p_2-q_5)$$
$$\times (2\pi)^4\delta^4(p_1+p_2-k_1-k_2)$$

$\hookrightarrow$ really still have to perform!

Counting: Call # externals + # internal vertices V

Call # lines (propagators) N

V each give a δ^4

N each give $\int d^4q$

but one δ^4 is the overall energy-mom conservation.

Only $(V-1)$ of the δ's eliminate $\int$'s

of $\int d^4q$'s — called $L = 1 + N - V$

Called "# of loops." Corresponds to graph thy meaning.

But let's finish the calc. we set off to do!

$$d\sigma = \frac{1}{2P_1^0 2P_2^0 |v_1 - v_2|} \int \frac{d^3k_1 \, d^3k_2}{(2\pi)^6 2k_1^0 2k_2^0} \lambda^2 (2\pi)^4 \delta^4(P_1 + P_2 - k_1 - k_2)$$

Let's introduce Mandelstam variables

$$s \equiv (P_1 + P_2)^2 = (\text{CM energy})^2 = (k_1 + k_2)^2$$

$$t \equiv (P_1 - k_1)^2 = (P_2 - k_2)^2$$

$$u \equiv (P_1 - k_2)^2 = (P_2 - k_1)^2$$

not all independent: $\cancel{s+t+u = \frac{1}{2}\{ (P_1+P_2)^2 + (k_1+k_2)^2 + (P_1-k_1)^2 + (P_2-k_2)^2 + (P_1-k_2)^2 + \}}$

$$\cancel{= \frac{1}{2}\{ P_1^2 + P_2^2}$$

$$s+t+u = 3P_1^2 + P_2^2 + k_1^2 + k_2^2 + 2P_1 \cdot \underbrace{(P_2 - k_1 - k_2)}_{= -P_1} = P_1^2 + P_2^2 + k_1^2 + k_2^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

(if all m's different...)

But this combination is symmetric and convenient.

$s > 0$. t, u usually < 0 (always if masses all equal)

Generally $\mathcal{M} = \mathcal{M}(s,t,u)$ as these are available invariants.

for us, $\mathcal{M} = \lambda$ and we are lucky

Integrals: use CM frame

$p_1^0 = E,\ \vec{p}_1 = \hat{z}\sqrt{E^2-m^2}\quad \vec{p}_2 = -\vec{p}_1\quad |v_1 - v_2| = 2|\vec{p}|/p^0$

$$\frac{\lambda^2}{8E\sqrt{E^2-m^2}} \int \frac{d^3k}{(2\pi)^2 4E^2}\, \delta(2k^0 - 2p^0)$$

I used $\delta^3(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2)$ to do $\int d^3\vec{k}_2$ and set $\vec{k}_2 = -\vec{k}_1$ & $k_2^0 = k_1^0$

$$\frac{\lambda^2}{128\pi^2 E^3\sqrt{E^2-m^2}} \underbrace{\int \cancel{k}\, \underbrace{k\,dk}_{=k^0dk^0}\, \delta(2k^0-2p^0)}_{\frac{E}{2}} \int d\Omega_k \quad = \int_0^{2\pi} d\phi \int d\cos\theta$$

$$= \frac{\lambda^2}{2^8\pi^2E^2} \int d\Omega_k \quad \text{~~Total cross section~~}$$

$$= \frac{\lambda^2}{2^6\pi^2 s}$$

Independent of angle - equal likelihood in each direction.

Total cross section: $\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = 4\pi$ BUT CAREFUL!

Final state $|k_1 k_2\rangle$ = final state $|k_2 k_1\rangle$.

We get differential σ to go to some angle & its opposite: $\vec{k}_1, \vec{k}_2 = -\vec{k}_1$

Same event as $-\vec{k}_1, +\vec{k}_1$.

Either put $\frac{1}{2}$ factor or integrate over $\frac{1}{2}$ of range.

$$\sigma = \frac{\lambda^2}{2^5 \pi s}$$

Remarks: σ = area = $1/\text{Energy}^2$ so $1/s$ expected.

Angle-independent - S-wave scattering.

$\times$ point vertex: particles must overlap to scatter.

Could have been as large as $8\pi/s$ (unitarity limit, π phase shift)

Result = $\left(\frac{\lambda}{16\pi}\right)^2 \times$ unitary limit ...