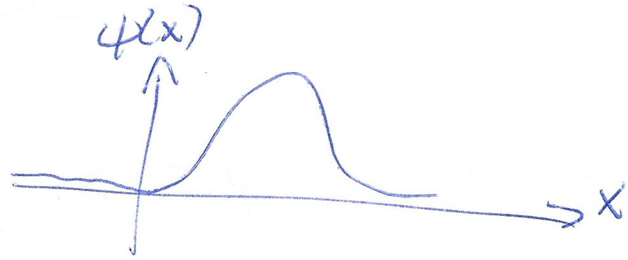
# Role of Lorentz symm in QFT



1) Operators - generators of Lorentz

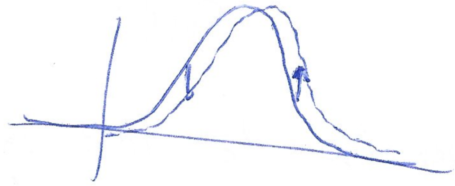
2) Possible field types - grown up version of possible spins in QM




Recall: translation op. in QM

$$P = -i(\hbar)\frac{\partial}{\partial x}$$

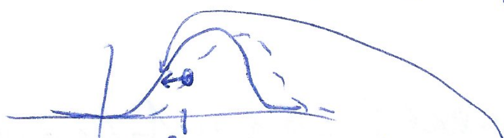
$$T(\Delta) = \exp(-i\Delta P/\hbar) \simeq 1 - \Delta\frac{\partial}{\partial x}$$



moves $\psi(x)$ forward

__Careful__ - if object moves forward, $x \to x + \Delta = x'$

Evaluator moves backward: $\psi(x) \to \psi'(x) = \psi(x - \Delta)$




2 perspectives: $\langle\psi_1|\mathcal{O}|\psi_2\rangle$

shift forward, $|\psi_2\rangle \to T(\Delta)|\psi_2\rangle$

$$\langle\psi_1|T^{-1}(\Delta)\,\mathcal{O}\,T(\Delta)|\psi_2\rangle$$

$$T^{-1}(\Delta) = T(-\Delta) = T^{+}(\Delta)$$

new state, forward?

or new op, backward?

$T^{-1}(\Delta)\,\mathcal{O}\,T(\Delta) = \mathcal{O}(x - \Delta)$

shift of ops backward.

Similarly I expect $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (\Lambda^\mu{}_\nu = \exp(\omega^\mu{}_\nu))$

has $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$ note, $\Lambda^{-1}x$ just like ...

~~Not~~ Will be implemented by some op's:

$\hat{p}$ : generator of translations. Infinites $\Delta$: $T(\Delta) = (1 - i\frac{p^\mu\Delta}{\hbar})$ if you want

$J$ : generator of rotations $\mathcal{U}(R) = (1 - i\frac{\vec{\Theta}\cdot\vec{J}}{\hbar}) + \mathcal{O}(\Theta^2)$

Lorentz version?

Careful: I know some field types will mix around.

$A^\mu$ (or any vector $V^\mu$):

$$V^\mu_{(x)} \xrightarrow{\text{Lorentz}} \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x)$$

the V's mix with each other

In general $\quad \Phi_a \to M_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x)$

some set of fields

matrix – generally reducible – on set of fields

Two Lorentz's in a row

$$\Phi_a \xrightarrow{\Lambda_1} \Phi'_a = M_{ab}(\Lambda_1)\Phi_b(\Lambda_1^{-1}x)$$

$$\Phi'_a \xrightarrow{\Lambda_2} \Phi''_a = M_{ab}(\Lambda_2)\Phi'_b(\Lambda_2^{-1}x)$$

$$= M_{ab}(\Lambda_2)M_{bc}(\Lambda_1)\Phi_c(\Lambda_1^{-1}\Lambda_2^{-1}x)$$

But what if I apply $\Lambda_2\Lambda_1$ in one step? After all, $\Lambda_2\Lambda_1$ also LT.

$$\Phi_a \xrightarrow{\Lambda_2\Lambda_1} I''_a = M_{ac}(\Lambda_2\Lambda_1)\Phi_c(\underbrace{\Lambda_1^{-1}\Lambda_2^{-1}}_{(\Lambda_2\Lambda_1)^{-1}=\Lambda_1^{-1}\Lambda_2^{-1}}x)$$

Therefore

$$M_{ab}(\Lambda_2)M_{bc}(\Lambda_1) = M_{ac}(\Lambda_2\Lambda_1)$$

Matrices M "represent" prod. rule of $\Lambda$'s.

Constrains possible (irreducible) matrices M. We will find all.

L14 P3

Think moment longer about rotations.

$\Theta_i$, $L_i$ or $J_i$ come from x-product - something special about 3D. Try to write WO x-prod.

Infinitesimal Rotation $R_{ij} = \delta_{ij} + \omega_{ij}$ $\omega$ small, $\omega_{ij} = -\omega_{ji}$.

$\omega_{ij} = \epsilon_{ijk}\Theta_k$

$J_{ij} = \epsilon_{ijk} J_k$

$$\begin{cases} \Theta_i \equiv \frac{\epsilon_{ijk}}{2}\omega_{jk} \\ J_i = \frac{\epsilon_{ijk}}{2} J_{jk} \end{cases}$$

why $\frac{1}{2}$? $\Theta_1 = \omega_{23}$. But $\epsilon\omega = \omega_{23} - \omega_{32} = 2\omega_{23}\ldots$

antisym tensor way to write rot. gen.

$$1 - i\frac{\Theta_i J_i}{\hbar} = 1 - \frac{i}{2}\frac{\omega_{ij} J_{ij}}{\hbar}$$ easy to show from above

$$[J_i, J_j] = i\hbar\epsilon_{ijk} J_k \quad \cancel{\Rightarrow [J_{ij}, J_{em}] = i\hbar(}$$

$$[V_i, J_j] = i\hbar\epsilon_{ijk} V_k \Rightarrow [V_i, J_{em}] = i\hbar(-g_{il}V_m + g_{im}V_l)$$

general vector (defines vector) show: $J_{em} = \epsilon_{ems} J_s \ldots$

$[J_{ij}, J_{em}]$ just transform each index ($ij$) like vector index by rule.

$$= i\hbar(g_{im}J_{lj} - g_{il}J_{mj} + g_{jm}J_{il} - g_{jl}J_{im})$$

Actually this follows from representation. Let's see how.

Remark: $1 - i\frac{\omega_{ij}}{2}J_{ij}$ = infinitesimal trans.

Finite = series of infinitesimal $\lim_{N\to\infty}\left(1 - \frac{i\omega_{ij}}{2N}J_{ij}\right)^N = \exp\left(-i\frac{\omega_{ij}J_{ij}}{2}\right)$

Product of 2 finites $R_2R_1 = \exp\left(-i\frac{\omega^2_{ij}}{2}J_{ij}\right)\exp\left(-i\frac{\omega^1_{ij}}{2}J_{ij}\right)$

1,2 are "which $\omega$".

$$= \left(1 - \frac{i\omega_2}{2N}J\right)^N \left(1 - \frac{i\omega_1}{2N}J\right)^N$$

To get $R_1R_2$, need to move $N$ past $N$. Need $\frac{1}{N^2}$ precision.

To know about commutation of Rotations, need commut. of infinites. rot. to $\mathcal{O}(1/N^2)$ or $(\text{infinitesimal})^2$.

Compare $R_2R_1$ to $R_1R_2$, $\omega_1, \omega_2$ infinitesimal

$$(R_1)_{ij} = \delta_{ij} + \omega_{1ij} + \tfrac{1}{2}\omega_{1ik}\omega_{1kj} \dots$$

$$(R_2)_{ij} = \delta_{ij} + \omega_{2ij} + \text{"}$$

$$(R_2R_1)_{ij} = (\delta_{ik} + \omega_{2ik} + \dots)(\delta_{kj} + \omega_{1kj} + \dots)$$

$$= \delta_{ij} + (\omega_2 + \omega_1)_{ij} + \left(\frac{\omega_2^2 + \omega_1^2}{2}\right) + (\omega_{2ik}\omega_{1kj})$$

$$(R_1R_2)_{ij} = \text{" \quad " \quad "} + (\omega_{1ik}\omega_{2kj})$$

note. not same

To order I need, $(R_2R_1)_{ij} = (R_1R_2)_{ij} + (\omega_{2ik}\omega_{1kj} - \omega_{1ik}\omega_{2kj})$ ) extra rotation.

note, $(\omega_2\omega_1 - \omega_1\omega_2)$ is antisymmetric ✓

$U_1 = (1 - \frac{i}{2}\omega_{1ij}\hat{J}_{ij} - \frac{1}{8}\omega^2 J^2)$ and $U_2 = (1 - \frac{i}{2}\omega_{2ij}\hat{J}_{ij})$ obey

$$U_2U_1 = U_1U_2 + \text{extra rotation} = (U_1U_2)\left(1 - \frac{i}{2}\left[\omega_{2ik}\omega_{1kj} - \omega_{1ik}\omega_{2kj}\right]\hat{J}_{ij}\right)$$

$$U_2U_1 = 1 - i(\omega_1 + \omega_2)_{ij}\hat{J}_{ij} - \frac{1}{8}(\omega_1^2 + \omega_2^2)J^2 + \left(-\frac{i}{2}\right)^2\omega_{2ij}\omega_{1\ell m}J_{ij}J_{\ell m}$$

$$U_1U_2 = \text{" \quad " \quad " \quad "} \quad \omega_{2ij}\omega_{1\ell m}J_{\ell m}J_{ij}$$ ) opposite order

Therefore $\left(-\frac{i}{2}\right)^2[J_{ij}, J_{\ell m}]\,\omega_{2ij}\omega_{1\ell m} = -\frac{i}{2}(\omega_{2rs}\omega_{1st} - \omega_{1rs}\omega_{2st})J_{rt}$

$$\omega_{2ij}\omega_{1\ell m}\left(-\frac{i}{2}\right)^2[J_{ij}, J_{\ell m}] = \omega_{2ij}\omega_{1\ell m}\left(-\frac{i}{2}\right)(\delta_{j\ell}J_{im} - \delta_{im}J_{\ell j})$$

for any antisymm $\omega_2, \omega_1$. Gives commutator we claimed!

Lorentz group version? Easy: $ij \to \mu\nu$

$$\mathcal{U}(\Lambda) = \exp\left(-i \frac{\omega_{\mu\nu}}{2} J^{\mu\nu}\right)$$

$J_{ij}$ : rotations

$J_{0i}$ : we'll see?!

Same representation argument:

$$[J^{\mu\nu}, J^{\alpha\beta}] = i(\pm g^{\nu\alpha} J^{\mu\beta} + g^{\mu\alpha} J^{\beta\nu} \pm g^{\mu\beta} J^{\alpha\nu} \mp g^{\nu\beta} J^{\mu\alpha})$$

sign due to opposite sign in metric: $g^{ij} = -\delta^{ij}$.

$J^{\mu\nu}$ $\mu,\nu = i,j$ rotation as before

$J^{0i}$ : generate boosts. Recall, for $\omega^{\mu}{}_{\nu} = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & & & \\ b_2 & & \cdots & \\ b_3 & & & \end{pmatrix}$

$$\exp\left(\frac{-i}{2\hbar} \omega^{\mu}{}_{\nu} J_{\mu}{}^{\nu}\right) = \text{boost}$$

$= \exp -i\,\omega_{0i} J^{0i}$ for $\mu = 0$ directly, for $\nu = 0$, play w. indices

$= \exp -i \vec{b} \cdot \vec{K}$ defines gen. of boosts

$K^i \equiv J^{0i}$ gen. of boosts. $\exp\left(\frac{-i}{2\hbar} \omega_{\mu\nu} J^{\mu\nu}\right) = \exp\left(-i(\theta_i J_i + b_i K_i)\right)$

Apply definitions & commutation relations: $J_i \equiv \epsilon_{ijk} J_{jk}$

$K_i \equiv J^{0i} = J_{i0}$

(as $J^{i0} = -J_{i0} = +J_{0i}$)

find:

$\to [J_i, J_j] = i\hbar \epsilon_{ijk} J_k$ $\vec{J}$ vector under rot.

~~$[J_i, K$~~

$[K_i, J_j] = i\hbar \epsilon_{ijk} K_k$ K is a vector under rotations no surprise.

$[K_i, K_j] = -i\hbar \epsilon_{ijk} J_k$ — sign!! Commute boosts to get a rotation! WTF ??

Thomas Precession...