Lorentz representation theory

So we saw that $[J_i, J_j] = i\epsilon_{ijk}J_k$
$[J_i, K_j] = i\epsilon_{ijk}K_k$
$[K_i, K_j] = -i\epsilon_{ijk}J_k$

Define $M_i = \frac{J_i + iK_i}{2}$ (also L_i) $N_i = \frac{J_i - iK_i}{2}$ (also R_i)

easy to show:
$[M_i, M_j] = i\epsilon_{ijk} M_k$
$[N, N] = i\epsilon N$
$[M, N] = 0$

splits in two!
But not quite product group:
$M_i^\dagger = N_i$, $N_i^\dagger = M_i$ not Hermitia

[$SO(4) \cong su(2)\times su(2)$ but $SO(3,1)$ not quite

Representation thy: most general IREP is
(Rep. of M) ⊗ (Rep. of N) $\left(\frac{m}{2}, \frac{n}{2}\right)$ is notation

$(0,0)$ $M_i = [0]$ $J_i = [0]$ Scalar
$N_i = [0]$ $K_i = [0]$

$(\frac{1}{2}, 0)$ $M_i = \left(\frac{\sigma_i}{2}\text{ spin}^{1/2}\right) \otimes [1] = \sigma_i/2$ $J_i = M_i + N_i = \sigma_i/2$ spin-1/2
$N_i = \begin{bmatrix}1&0\\0&1\end{bmatrix} \otimes [0] = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ $K_i = -iM_i + iN_i = -i\sigma_i/2$ huh??

Left-Weyl spinor

$(0, \frac{1}{2})$ $M_i = [0] \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ $J_i = \sigma_i/2$ spin-1/2.

$N_i = [1] \otimes \sigma_i/2 = \sigma_i/2$ $K_i = +i\sigma_i/2$ different than L

Right-Weyl spinor

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higher spins harder to understand.

if ψ_a transforms under rep Λ, it means $\exp(-i\vec{\theta}\cdot \vec{J} + i\vec{b}\cdot \vec{K})\psi$

$\psi_a \longrightarrow \exp\left(-i\theta_i\frac{\sigma_i}{2} + b_i\frac{\sigma_i}{2}\right)_{ab}\psi_b$

example: $(1,0)$

$$M_i = -i\epsilon_{i(jk)} \otimes [1] \qquad (J_i)_{jk} = -i\epsilon_{ijk} \quad \text{spin-1 rep.}$$

$$N_i = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix} \otimes [0] \qquad (K_i)_{jk} = -\epsilon_{ijk}$$

This is some spin-1 field. No spin-0 component.

What Lorentz structure can always be spin-1 ??

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \text{antisymm tensor} = \begin{array}{c c} & \begin{array}{cccc} \nu=0 & \nu=1 & \nu=2 & \nu=3 \end{array} \\ \begin{array}{c} \mu=0 \\ \mu=1 \\ \mu=2 \\ \mu=3 \end{array} & \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ +E_1 & 0 & -B_z & B_y \\ +E_2 & B_z & 0 & -B_x \\ +E_3 & -B_y & B_x & 0 \end{bmatrix} \end{array}$$

All entries are "spin-1". But 6, not 3.

But if $B_i = iE_i$ is enforced then 3 comp.

~~Also, in this case the rule~~

Transform rule means $\quad V_j \xrightarrow{\Lambda} \exp\begin{bmatrix} -i\theta_i(-i)\epsilon_{ijk} \\ +ib_i(-)\epsilon_{ijk} \end{bmatrix} V_k$

Rotation $\quad V_j \to V_j - \epsilon_{ijk}\theta_i V_k = V_j + \epsilon_{jik}\theta_i V_k \quad$ rotation V

Boost $\quad V_j \to V_j + i\epsilon_{ijk} b_i V_k \Rightarrow \vec{V} + i\vec{b} \times \vec{V}$

Well, under boost $\quad E \to E - \vec{b} \times \vec{B} \quad$ so if $\vec{B} = -i\vec{E}$ this works out!

$$B \to B + \vec{b} \times \vec{E}$$

$(1,0)$ is $F^{\mu\nu}$ if $\vec{B} = -i\vec{E}$ self-dual

$(0,1)$ is $F^{\mu\nu}$ if $\vec{B} = +i\vec{E}$ anti-selfdual

$F^{\mu\nu}$ is $(1,0) \oplus (0,1)$ so I can get $\vec{E}, \vec{B}$ independent.

Understand $(\frac{1}{2},0)$ & $(0,\frac{1}{2})$ and develop notation

Field trans. under $(\frac{1}{2},0)$: $\psi^\alpha \xrightarrow{\Lambda} \psi'^\alpha(\Lambda^{-1}x) = S^\alpha{}_\beta(\Lambda)\psi^\beta(x)$

$$S^\alpha{}_\beta = \exp\left[(-i\theta_i + b_i)\frac{\sigma_i}{2}\right]^\alpha{}_\beta$$

Return to

$\mathbb{C}$ matrix. ψ^α must be $\mathbb{C}$ field. $(\psi^\alpha)^\dagger \equiv \psi^{\dot\alpha}$ also exists.

$\mathcal{L} = \mathcal{L}(\psi, \psi^\dagger)$ both as $\mathcal{L}$ must be Hermitian

Also $\mathcal{L}$ must be scalar, and spin-½ need even # to build scalar.

Must think about how to combine ψ^α, $\psi^{\dot\alpha}$'s.

$$(\psi^\alpha)^\dagger \longrightarrow \exp\left((+i\theta_i+b_i)\frac{\sigma_i^\dagger}{2}\right)^{\dot\alpha}{}_{\dot\beta}\,\psi^{\dot\alpha}$$ is some new rep? NO

~~Define $\epsilon_{\dot\alpha\dot\beta} = \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$. Note: $\epsilon\sigma = -\sigma^*\epsilon$~~

~~because secretly $\epsilon = -i\sigma_2$. $\sigma_2\sigma_2 = \sigma_2\sigma_2$ but $\sigma_2^* = -\sigma_2$~~

~~but $\sigma_2\sigma_1 = -\sigma_1\sigma_2$ $\quad \sigma_1^* = \sigma_1$~~

~~$\sigma_2\sigma_3 = -\sigma_3\sigma_2$ $\quad \sigma_3^* = \sigma_3$~~

How can you combine spin-½ fields?

After all, we need $\mathcal{L} \longrightarrow$ scalar: even # of spin-½ to get a scalar

well, for ψ_1^a, ψ_2^b spin ½, we know $\epsilon_{ab}\psi_1^a\psi_2^b = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ is spin-0

Is it Lorentz scalar?

$$\epsilon_{ab}\psi_1^a\psi_2^b \xrightarrow{\Lambda} \epsilon_{ab}\exp\left((-i\theta+b)\cdot\frac{\sigma}{2}\right)^a{}_{a'}\exp\left((-i\theta+b)\cdot\frac{\sigma}{2}\right)^b{}_{b'}\psi_1^{a'}\psi_2^{b'}$$

$$\epsilon_{ab}\,\psi_1^a\psi_2^b \longrightarrow \left(\epsilon_{ab}\, e^{-(i\theta\mp b)\frac{\sigma}{2}\,a}{}_{a'}\left(e^{-(i\theta\mp b)\frac{\sigma}{2}}\right)^b{}_{b'}\right)\psi_1^{a'}\psi_2^{b'}$$

$\epsilon\sigma = -\sigma^*\epsilon$ try it: $\sigma_2\epsilon = \epsilon\sigma_2$ but $\sigma_2^* = -\sigma_2$

$\sigma_1\epsilon = -\epsilon\sigma_1$ but $\sigma_1 = \sigma_1^*$

$\sigma_3\epsilon = -\epsilon\sigma_3$ but $\sigma_3 = \sigma_3^*$

$$\epsilon_{ab}\, e^{(-i\theta\mp b)\frac{\sigma}{2}\,b}{}_{b'} = e^{(+i\theta\mp b)\frac{\sigma^*}{2}}{}_a{}^b\,\epsilon_{bb'} = \epsilon_{bb'}\left[e^{(i\theta\mp b)\frac{\sigma}{2}}\right]^b{}_a$$

as $(\sigma^*)^T = \sigma^\dagger = \sigma$

But $\left(e^{+(i\theta\mp b)\frac{\sigma}{2}}\right)^b{}_a\left(e^{-(i\theta\mp b)\frac{\sigma}{2}}\right)^a{}_{a'} = \delta^b{}_{a'}$ they cancel!

$\epsilon_{a'b'}\,\psi_1^{a'}\psi_2^{b'}$ r unchanged. this is a scalar!

Oh, and I can combine 2 spin-1/2 together 3 other ways, which should form a vector:

$\epsilon_{ab}\,\sigma_i{}^b{}_{b'}\,\psi_1^a\psi_2^{b'}$ transforms as vector

and as $(1,0)$ under Lorentz (not shown)

But I cannot get a 4-vector this way.
Need both R, L fields. How are they related?

~~Go back to~~

Now consider $\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\,\psi^{\dot\beta} \longrightarrow \epsilon_{\dot\alpha\dot\beta}\, e^{(+i\theta+b)\frac{\sigma}{2}\,\dot\beta}{}_{\dot\gamma}\,\psi^{\dot\gamma}$

$$= e^{(-i\theta\mp b)\frac{\sigma}{2}}\,\epsilon\psi \qquad \psi_{\dot\alpha} \text{ is in R rep.}$$

Ah - the R-rep

SO $\psi^\alpha \xrightarrow{\Lambda} \left(e^{(-i\theta+b)\cdot\frac{\sigma}{2}}\right)^\alpha{}_\beta\, \psi^\beta \equiv S^\alpha{}_\beta\, \psi^\beta$

$\psi_{\dot\alpha} \xrightarrow{\Lambda} \left(e^{(-i\theta - b)\cdot\frac{\sigma}{2}}\right)_{\dot\alpha}{}^{\dot\beta}\, \psi_{\dot\beta} = S_{\dot\alpha}{}^{\dot\beta}\, \psi_{\dot\beta}$

are related by $\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}(\psi^\beta)^\dagger$

$\psi_\alpha \psi_\beta$ can combine via $\epsilon^{\alpha\beta}$ into scalar $(0,0)$

via $\epsilon\vec\sigma$ into antisymm tensor $(1,0)$

what about $\psi_{1\dot\alpha}\psi_2^\alpha$? How can I combine them?

Define $\sigma^\mu_{\dot\alpha\alpha} = \begin{pmatrix} \delta_{\dot\alpha\alpha} \\ +\sigma^x_{\dot\alpha\alpha} \\ +\sigma^y_{\dot\alpha\alpha} \\ +\sigma^z_{\dot\alpha\alpha} \end{pmatrix}$. Claim: $\psi_1^{\dot\alpha}\sigma^\mu_{\dot\alpha\alpha}\psi_2^\alpha$

$$\Big\downarrow \Lambda$$

$\cancel{\psi_{1\dot\alpha}\psi_2^\alpha\left(\epsilon^{\dot\alpha\dot\beta}S_{\dot\beta}{}^{\dot\gamma}\sigma^\mu_{\dot\gamma\gamma}S^\gamma{}_\alpha\right)}$ $\cancel{\Lambda^\mu{}_\nu\sigma^\nu}$

$\psi_1^{\dot\alpha}\psi_2^\alpha \underbrace{\left(S^{\dot\beta}{}_{\dot\alpha}\,\sigma^\mu_{\dot\beta\beta}\,S^\beta{}_\alpha\right)}_{\Lambda^\mu{}_\nu\sigma^\nu_{\dot\alpha\alpha}}$

transforms like a 4-vector.

In particular $\psi_{1\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\delta_{\dot\beta\beta}\psi^\beta = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

is a rotation-scalar, but NOT Lorentz scalar.

Similarly I can rewrite $\psi_1^{\dot\alpha}\sigma^\mu_{\dot\alpha\alpha}\psi_2^\alpha = \psi_{1\dot\beta}\,\epsilon^{\dot\beta\dot\alpha}\sigma^\mu_{\dot\alpha\alpha}\epsilon^{\alpha\beta}\psi_{2\beta}$

now $-\epsilon\,\mathbb{1}\,\epsilon = \mathbb{1}$

$-\epsilon\,\sigma\,\epsilon = -\sigma^T$

$= \psi_{2\beta}\,\bar\sigma^{\mu\beta\dot\alpha}\psi_{1\dot\alpha}$ $\quad \bar\sigma^\mu = \begin{pmatrix} \delta^{\beta\dot\alpha} \\ -\sigma \\ -\sigma \\ -\sigma \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ -\vec\sigma \end{pmatrix}$

Better notation: γ-matrices

Suppose a thy. has two fields, ξ^α and $\eta_{\dot\alpha}$
which for fun I choose to write as L, R respectively.
Note that $\xi_{\dot\alpha}$, η^{α} also exist & Lagrangian must be written in terms of both.

Define $\psi = \begin{bmatrix} \eta_{\dot\alpha} \\ \xi^\alpha \end{bmatrix}$ 4-component.

$\bar\psi = [\xi^{\dot\alpha} \quad \eta_\alpha]$ note index locations.

$= \psi^\dagger \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Then $\bar\psi\psi = \xi^{\dot\alpha}\eta_{\dot\alpha} + \eta_\alpha\xi^\alpha$ is Hermitian & scalar

$\bar\psi\gamma^\mu\psi$, $\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$ is 4-vector

(or $\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{bmatrix}$)

note, $\frac{i}{4}[\gamma^\mu, \gamma^\nu] = [M^{\mu\nu}]$ give the things which generate Lorentz:

$S(\Lambda) = \exp\left(-\frac{i}{2}\Omega_{\mu\nu}\frac{i}{4}[\gamma^\mu, \gamma^\nu]\right)$.

$\frac{i}{2}\bar\psi[\gamma^\mu, \gamma^\nu]\psi$ is $(1,0)\otimes(0,1)$ rep

Easy way to package Hermitian version of all Lorentz comb. I can build.