~~Dirac Equation~~

Last remarks about γ-matrices

1) Sometimes I only have 1 2-comp. field (Standard Model)
(neutrinos)

Then define $\psi = \begin{bmatrix} \xi_a \\ \xi^{\dagger \dot{a}} \end{bmatrix}$ same field top & bottom

Some possible terms now equivalent, must be careful. Majorana
(Notation) (field)

2) I am always free to make
Similarity Transformation $\psi \to S\psi$
(basis change)

$\bar{\psi} \to \cancel{\psi^\dagger S^\dagger \gamma^0}\ \bar{\psi} S^{-1}$

$\gamma^\mu \to S \gamma^\mu S^{-1}$ S unitary 4x4

gives all same physics.
Our basis: Chiral basis - best for very relativistic settings
Other bases better, eg, for $v \ll 1$.

In all cases, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{1}_{4\times4}$ "Clifford Algebra"

enough to derive all Lorentz properties.

Let's build a Lagrangian. Consider 1 Dirac field

$$\mathcal{L} = -m\underbrace{\bar{\psi}\psi}_{\text{scalar}} + i\,\underbrace{\bar{\psi}\gamma^\mu\partial_\mu\psi}$$

as $\bar{\psi}\gamma^\mu\psi$ 4-vector, cancels 4-vector index of ∂_μ

Hermitian? $(\psi)^* = \psi^\dagger$, Recall, $\bar{\psi} = \psi^\dagger\gamma^0$. Oh yea, $\gamma^{0\dagger} = \gamma^0$

$\gamma^{i\dagger} = -\gamma^i$

$\gamma^0\gamma^{i\dagger} = \gamma^i\gamma^0$

(in general $\gamma^0\gamma^{\mu\dagger} = \gamma^\mu\gamma^0$)

$\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$

$$(\bar{\psi})^* = \gamma^{0\dagger}\psi = \gamma^0\psi$$

$$(\bar{\psi}\psi)^\dagger = (\psi^\dagger\gamma^0\psi)^\dagger$$

$$\psi^\dagger\gamma^0\psi = \bar{\psi}\psi \text{ is Hermitian ✓}$$

$$\bar{\psi}\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\psi$$

$$(\bar{\psi}\gamma^\mu\psi)^\dagger = \psi^\dagger\gamma^{\mu\dagger}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi \text{ Hermitian}$$

But

$$A\partial_\mu B = -B\partial_\mu A + \underbrace{\partial_\mu(AB)}_{\text{total deriv.}}$$

$$(\bar{\psi}\gamma^\mu\partial_\mu\psi)^\dagger = (\partial_\mu\bar{\psi})\gamma^\mu\psi = -\bar{\psi}\gamma^\mu\partial_\mu\psi + \text{tot deriv.}$$

Hence I need an i in that part of $\mathcal{L}$.

Technically I should probably write $\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right)$

$$\equiv i\,\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}_\mu\psi$$

$$A\overleftrightarrow{\partial}_\mu B \equiv \frac{1}{2}\left(A\partial_\mu B - (\partial_\mu A)B\right)$$

Some people do this & it can matter in curved space. But let's not get fussy.

It could be $-m'\bar{\psi}\psi + Ai\bar{\psi}\gamma^\mu\partial_\mu\psi$ but then $\psi_{new} = \sqrt{A}\,\psi$

$m_{new} = m'/A$ .... get rid of A.

Dimensions: $[\psi] = 3/2$

Interacting theory? Since $[\psi] = 3/2$, not much room to build dim ≤ 4 objects!

$$\mathcal{L} = -m\bar{\psi}\psi + i\bar{\psi}\gamma^\mu\partial_\mu\psi + y\varphi\bar{\psi}\psi + eA_\mu\bar{\psi}\gamma^\mu\psi$$

appear to be only possibilities (for one ψ. Could have, eg,

$$\varphi_a\bar{\psi}_i T^a_{ij}\psi_j, \quad A^a_\mu\bar{\psi}_i T^a_{ij}\psi_j$$

for multi-comp φ, A, ψ)

$y\varphi\bar{\psi}\psi$: "Yukawa coupling"

$eA_\mu\bar{\psi}\gamma^\mu\psi$: "gauge coupling" same as $i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi$ turns out to be better way to understand it.

Let's look at free thy and ask (possibly stupid) question "What's classical field dynamics of ψ field??"

$$\mathcal{L} = -m\bar{\psi}\psi + i\bar{\psi}\gamma^\mu\partial_\mu\psi$$

Apply $\frac{\delta}{\delta\bar{\psi}(x)} S = 0 \longrightarrow -m\psi + i\gamma^\mu\partial_\mu\psi = 0$ Dirac Equation

$(i\gamma^\mu\partial_\mu - m)\psi = 0$ what are solutions?

Well they are also solutions to $(i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi = 0$

$[-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2]\psi = 0$

Oh, $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$ so I can $\gamma^\mu\gamma^\nu \to \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu}$

$(-\partial_\mu\partial^\mu - m^2)\psi = 0$ Klein-Gordon equation

Momentum-space: $(p^2 - m^2)\psi = 0$ or $E^2 = \vec{p}^2 + m^2$ good.

So ψ must obey $\psi(x) = \int d^4p \, e^{ip\cdot x} \psi(p)$

with $(p^2 - m^2)\psi(p) = 0$

But that's necessary, not sufficient constraint on $\psi(p)$.

Applying $(i\partial^\mu \partial_\mu + m)$ may have sneaked in spurious solutions.

~~But before trying to do better, let's ask about interacting w EM field, eg, $A^\mu \neq 0$~~

$$\cancel{\left(i(\partial^\mu - ieA^\mu)\gamma_\mu + m\right)\left(i(\partial^\nu - ieA^\nu)\gamma_\nu - m\right)\psi = 0}$$

$$\cancel{\left[-\gamma^\mu\gamma^\nu \underbrace{(\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu)}_{\text{do not commute.}} - m^2\right]\psi = 0}$$