Let's look at interaction with classical EM field

$$\mathcal{L}_{Dirac} = \bar{\Psi}\left(i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right)\psi$$

Dirac Eq: $0 = [i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\psi$

Multiply by $[i\gamma^\nu(\partial_\nu - ieA_\nu) + m]$:

$$0 = \left[-\gamma^\mu\gamma^\nu\underbrace{(\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu)} - m^2\right]\psi$$

Call $\partial_\mu - ieA_\mu = D_\mu$.

$$D_\mu D_\nu = \underbrace{\tfrac{1}{2}\{D_\mu, D_\nu\}}_{\mu,\nu \text{ symm}} + \underbrace{\tfrac{1}{2}[D_\mu, D_\nu]}_{\mu,\nu \text{ antisymm}}$$

$$\Big(\underbrace{-\tfrac{1}{4}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{= 2g^{\mu\nu}}\{D_\mu, D_\nu\}}_{-D_\mu D^\mu} \underbrace{- \tfrac{1}{4}[\gamma^\mu,\gamma^\nu][D_\mu, D_\nu]}_{??} - m^2\Big)\psi = 0$$

same as for scalar

$$[D_\mu, D_\nu] = -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = -ieF_{\mu\nu} \quad \text{aha!}$$

and $\frac{1}{4}[\gamma^\mu,\gamma^\nu]$? consider $\mu=0, \nu=i$: $\frac{1}{4}\left(\begin{bmatrix}0 & 1\!\!1\\ 1\!\!1 & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix} - \begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix}\begin{bmatrix}0 & 1\!\!1\\ 1\!\!1 & 0\end{bmatrix}\right)$

$$= \frac{1}{4}\begin{bmatrix}-2\sigma_i & 0\\ 0 & 2\sigma_i\end{bmatrix} = \begin{bmatrix}-\sigma_i/2 & 0\\ 0 & \sigma_i/2\end{bmatrix}$$

while for $\mu=i$, $\nu=j$: $\frac{1}{4}\begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{bmatrix} - \begin{bmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix}[\sigma_j,\sigma_i] & 0\\ 0 & [\sigma_j,\sigma_i]\end{bmatrix} = \frac{-i}{2}\epsilon_{ijk}\begin{bmatrix}\sigma_k/2 & 0\\ 0 & \sigma_k/2\end{bmatrix} = \epsilon_{ijk}\begin{bmatrix}-\frac{i\sigma_k}{2} & 0\\ 0 & -\frac{i\sigma_k}{2}\end{bmatrix}$$

Recall, $J_i = \frac{\sigma_i}{2}$

$K_i = +\frac{i\sigma_i}{2}$ ~~Left~~ Right

$-\frac{i\sigma_i}{2}$ Left

so $\frac{i}{4}[\gamma^\mu,\gamma^\nu] = \epsilon^{ijk}\frac{\sigma_k}{2}$ spin

$\frac{i\sigma}{2}$ for $(i0)$ comp K

$-\frac{i\sigma}{2}$ are Lorentz generators

$\frac{1}{4}[\gamma^\mu,\gamma^\nu] = -i\mathcal{M}^{\mu\nu}$ spin gen.

Dirac Eq $\left(-D_\mu D^\mu - \underbrace{(-iM^{\mu\nu})(-ieF_{\mu\nu})}_{} - m^2\right)\psi = 0$

$M^{\mu\nu}F_{\mu\nu} = 2\underbrace{\vec{S}\cdot\vec{B}}_{\text{spin-mag interaction}} + 2\underbrace{\vec{K}\cdot\vec{E}}_{\text{some required relativistic generalization bit.}}$

$p^2 - 2\vec{S}\cdot\vec{B} - m^2 = 0$

$E^2 = p^2 + m^2 + 2\vec{S}\cdot\vec{B}$, $E \simeq m + \frac{p^2}{2m} + \frac{2\vec{S}\cdot\vec{B}}{2m}$ in nonrel. limit

gyroscopic ratio $g=2$

What if $A^0 = \frac{e\,\cancel{\ }Z}{4\pi r}$, $\vec{A}=0$, eg. Hydrogen atom?

Same reasoning, $\left(-(\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu) - m^2 + ie\begin{bmatrix}\vec{E}\cdot\vec{\sigma} & 0\\ 0 & -\vec{E}\cdot\vec{\sigma}\end{bmatrix}\right)\psi = 0$

extra part like B-field
Note, sign opposite for....

now look for energy eigenstates

$\partial_0\psi = -i\omega\psi$. ~~$\partial \ldots$~~

$(\partial_0 - ieA_0)\psi = -i(\omega + \frac{\alpha Z}{r})\psi$

$\partial_i - ieA_i = \partial_i$ only (whew!)

and $\vec{E} = \frac{Z\alpha\hat{r}}{r^2}$ of course.

$$\left[\left(\omega + \frac{\alpha Z}{r}\right)^2 - m^2 + \nabla^2 + \begin{bmatrix} i & 0\\ 0 & -i\end{bmatrix}\frac{Z\alpha\,\vec{\sigma}\cdot\hat{r}}{r^2}\right]\psi = 0$$

And as usual, $\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r - \frac{\vec{L}^2}{r^2}$

so, $(\omega^2 - m^2)\psi = \underbrace{\left(-\partial_r^2 - \frac{2}{r}\partial_r + \left(L^2 - Z^2\alpha^2 + \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}Z\alpha\,\vec{\sigma}\cdot\hat{r}\right)\frac{1}{r^2} - \frac{2Z\alpha\omega}{r}\right)}_{\text{Call this } H.}\psi$

Assume $\psi = R(r)F(\theta,\varphi)$: F may be matrix

Now $[L,H]\neq 0$. But $[(L+\frac{\sigma}{2})^2, H]=0$ ($L+\frac{\sigma}{2} = J$ of course, total ang mom)

and $[L^2, L+\frac{\sigma}{2}]=0$

so seek eigenvectors of $\hat{J}^2 = j(j+1)$

will be mix of $l = j+1/2$ (l_+), $l = j-1/2$ (l_-). Work in 2-comp basis of these l-values. (in each 2-comp. block!)

$$\langle l_+|\vec{\sigma}\cdot\hat{r}|l_+\rangle = 0 = \langle l_-|\vec{\sigma}\cdot\hat{r}|l_-\rangle$$

But $(\sigma\cdot\hat{r})^2 = \mathbb{1}$ so $\begin{bmatrix}\langle l_+| & \langle l_-|\end{bmatrix}\vec{\sigma}\cdot\hat{r}\begin{bmatrix}|l_+\rangle \\ |l_-\rangle\end{bmatrix} = \begin{bmatrix}0 & \pm 1\\ \pm 1 & 0\end{bmatrix}$

$$L^2 - Z^2\alpha^2 \mp iZ\alpha\vec{\sigma}\cdot\vec{r} = \begin{bmatrix}(j+\frac12)(j+\frac32) - Z^2\alpha^2 & \mp iZ\alpha \\ \mp iZ\alpha & (j-\frac12)(j+\frac12) - Z^2\alpha^2\end{bmatrix}$$

has eigenvalues $\lambda(\lambda+1)$ with $\lambda = \sqrt{(j+\frac12)^2 - Z^2\alpha^2}$

$\lambda = \quad " \quad -1$

eg, eigenvalue $= \sqrt{}(\sqrt{}+1)$ or $\sqrt{}(\sqrt{}-1)$.

Substitute this eigenvalue for H and try to solve radial problem

$$(\omega^2 - m^2)R(r) = \left(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{\lambda(\lambda+1)}{r^2} - \frac{2Z\alpha\omega}{r}\right)R(r)$$

Almost-Schrödinger radial eq. Solutions $n = 1, \dots$ with $j = \frac12, \frac32, \dots, n-\frac12$

Solution: defining $\delta_j = j + \frac12 - \sqrt{(j+\frac12)^2 - Z^2\alpha^2}$ $\left(\simeq \frac{Z^2\alpha^2}{2j+1}\right)$

$$\omega_{nj} = \frac{m}{\sqrt{1 + (Z^2\alpha^2/(n-\delta_j)^2)}} = \underset{\substack{\uparrow \\ \omega = m \\ \text{duh}}}{m} - \underset{\substack{\uparrow \\ \text{Schrö.}}}{\frac{mZ^2\alpha^2}{2n^2}} \underbrace{- \frac{mZ^4\alpha^4}{n^3(2j+1)} + \frac{3}{8}\frac{mZ^4\alpha^4}{n^4}}_{\underline{\text{correct}}\text{ fine structure}} + \mathcal{O}(\alpha^6)$$

something new

1) Now I need to go back and find actual radial, spin structures of solutions... straightforward

2) If $Z\alpha = 1$, $\delta_j = 1$, $n - \delta_j \geq 0$, $E \to 0$, something terrible happens. That takes nucleus with $Z = 1/\alpha = 137$. Ha! We're only up to 118...

3) Note that solutions not pure ℓ-eigenstates. Spin-orbit mixing.

4) Gets fine structure right. Hyperfine wrong. That requires considering real live fluctuating quantum EM field, not $A^0 = \frac{-eZ}{4\pi r}$ ....

Lets get down & dirty with finding spin-state stuff.

We saw free ψ: Dirac eq.

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Assume $\psi(x,t) = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip\cdot x} \times \begin{pmatrix} u(p) \text{ if } p^0 > 0 \\ V(p) \text{ if } p^0 < 0 \end{pmatrix}$

$(p^0 = \pm\sqrt{\vec{p}^2 + m^2})$

I know so far that Dirac eq forces $\uparrow$. But it also constrains the 4-comp. column u // v which can appear at a given p. What can that be?

$i\partial^\mu \gamma_\mu \to p^\mu \gamma_\mu$. So $(p^\mu\gamma_\mu - m) u = 0 \quad p^0 = +\sqrt{\vec{p}^2 + m^2}$

$\qquad\qquad\qquad\qquad$ " $\quad V = 0 \quad p^0 = -\sqrt{\vec{p}^2 + m^2}$

$$p^\mu\gamma_\mu - m = \begin{bmatrix} -m & 0 & E+p_z & p_x - ip_y \\ 0 & -m & p_x + ip_y & E - p_z \\ E - p_z & -p_x + ip_y & -m & 0 \\ -p_x - ip_y & E + p_z & 0 & -m \end{bmatrix}$$

This is indeed a degenerate matrix.

What are the zero-eigenvectors?

Those are the allowed u // V.

Looks like real work. Let's take a short-cut.

Consider at-rest: $p^\mu \gamma_\mu - m = m \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ for $p^0 > 0$

two zero- and two nonzero-eigenvalues. Two e-vectors w. zero eigenvalue

~~Also~~ two eigen-vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$: $u_+(p), u_-(p)$

spin along/against z.

for $p^0 < 0$, $p^\mu \gamma_\mu - m = m \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$ e-vectors $V_+(p) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

$V_-(p) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

Normalization? Mult. each by $\sqrt{m}$ because ψ is dim $3/2$ not dim 1

For $p_z \neq 0$ only, we can seek a solution

$$p^\mu \gamma_\mu - m = m \begin{bmatrix} -1 & 0 & (1+\beta)\gamma & 0 \\ 0 & -1 & 0 & (1-\beta)\gamma \\ (1-\beta)\gamma & 0 & -1 & 0 \\ 0 & (1+\beta)\gamma & 0 & -1 \end{bmatrix} \quad u_+ = \sqrt{\gamma m} \begin{bmatrix} \sqrt{1+\beta} \\ 0 \\ \sqrt{1-\beta} \\ 0 \end{bmatrix} \quad u_- = \sqrt{\gamma m} \begin{bmatrix} 0 \\ \sqrt{1-\beta} \\ 0 \\ \sqrt{1+\beta} \end{bmatrix}$$

for large velocity $\beta \to 1$, $u_+ \to \sqrt{2E} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad u_- \to \sqrt{2E} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

eg. $\vec{V} \parallel \vec{S}$ is purely upper (R-handed)

$\vec{V}$ anti $\vec{S}$ purely lower (L-handed)

Similarly $\quad V_+ = \sqrt{\gamma m} \begin{bmatrix} \sqrt{1-\beta} \\ 0 \\ -\sqrt{1+\beta} \\ 0 \end{bmatrix} \quad V_- = \sqrt{\gamma m} \begin{bmatrix} 0 \\ \sqrt{1+\beta} \\ 0 \\ -\sqrt{1-\beta} \end{bmatrix}$

For any other direction $\hat{p}$

1) find 2-spinor which is + or − along $\hat{p}$ direction, eg

$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ for P_x only...

Use this instead of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

Other cheap approach:

Take $\vec{p}=0$ solution: Rotate and/or boost

$$\exp(+i\vec{b}\cdot\vec{K}) = \begin{bmatrix} e^{b/2} & 0 & 0 & 0 \\ 0 & e^{-b/2} & 0 & 0 \\ 0 & 0 & e^{-b/2} & 0 \\ 0 & 0 & 0 & e^{+b/2} \end{bmatrix}$$

for b_z

from explicit form of boost

note, $e^{b/2} = \sqrt{\gamma^2}\sqrt{1+\beta^2} = \left[\frac{1+\beta}{1-\beta}\right]^{1/4}$??

$e^{2b} = \frac{1+\beta}{1-\beta}$ solve for $\beta = \frac{e^{b/2}-e^{-b/2}}{e^{b/2}+e^{-b/2}}$

$\beta = \tanh b$ ✓

Normalization choice:

$\bar{V}V = -2m$ negative

$V^\dagger V = +2E$ positive

$\bar{u}u = 2m$ − scalar

$u^\dagger u = 2E$ − time comp of 4-vector

$(u^\dagger\gamma^\mu u = 2p^\mu)$

Well technically, each $\times\, \delta_{\sigma\sigma'}$ that it's the same u.

Completeness: $\sum_\sigma u_\sigma(p)\bar{u}_\sigma(p) = \gamma^\mu p_\mu + m$

$\sum_\sigma V\ \bar{V} = \gamma^\mu p_\mu - m$

Name $A_\mu\gamma^\mu \equiv \not{A}$ slash notation (sorry)