Quantizing ψ

Let's <u>guess</u> that Hilbert space = $\{\mathbb{C}$ func's over classical values $\psi(x)\}$

as in scalar case.

If ψ is my gen coord Q, then $P_\mu = \frac{\delta \mathcal{L}}{\delta \partial^\mu \psi} = i\bar{\psi}\gamma_\mu$

in particular, canon. momentum $P_0 = \pi = i\bar{\psi}\gamma_0 = i\psi^\dagger$

so I <u>expect</u> $[\psi^\dagger_a(x), \psi_b(y)] = -\delta^3(x-y)\delta_{ab}$ note sign!

$= -i[\pi, \psi] = -i(-i\ldots)$"

and $[\psi_a(x), \psi_b(y)] = 0 = [\psi^\dagger_a(x), \psi^\dagger_b(y)]$ Sounds reasonable (?)

Then write

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\vec{p}\cdot\vec{x}} \sum_{\sigma=+,-} \left(\hat{b}_{p\sigma} u_\sigma(p) + \hat{d}^\dagger_{-p\sigma} V_\sigma(-p)\right)$$

$\hat{b}, \hat{d}$ some <u>operators</u> w. same dimension as $a, a^\dagger$ for scalars

u, V are 4-comp spinors - columns of #5 - which give complete basis of components & which I expect to be best way to incorporate what we already know about sol'ns

so

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-i\vec{p}\cdot\vec{x}} \sum_\sigma \left(\hat{b}^\dagger_{p\sigma} u^\dagger_\sigma(p) + \hat{d}_{-p,\sigma} V_\sigma(-p)\right)$$

$H =$ "$p\dot{q} - \mathcal{L}$"

$$= \int d^3x \left(i\psi^\dagger \partial_0 \psi - i\bar{\psi}\gamma^\mu \partial_\mu \psi + m\bar{\psi}\psi\right)$$

$= i\bar{\psi}\gamma^0\partial_0\psi$ kills ↑ term here

$$H = \int d^3x \left(-i\bar{\psi}\,\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi\right)$$ huh - no time deriv's ??

Commutation relations of $\psi, \psi^\dagger$:

long story short, $[\psi, \psi] = 0$

$$[\psi^\dagger, \psi] = -\delta^3(x-y)\,\delta_{\alpha\beta}$$

$$[\psi^\dagger, \psi^\dagger] = 0$$

equivalent to

$$[b_\sigma(p), b^\dagger_{\sigma'}(q)] = (2\pi)^3 2E_p\, \delta^3(\vec{p}-\vec{q})$$

$$[b, d^\dagger] = [b, d] = [b, b] = \ldots = 0$$

$$[d, d^\dagger] = (2\pi)^3 2E_p\, \delta^3(p-q)$$

Assume (the latter), you can show (the former). Completeness - only way it could work out.

Insert our expressions into H

$$H = \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6 2p^0 2q^0} \left(e^{-i\vec{q}\cdot\vec{x}}\right)\left(\sum_{\sigma'} b^\dagger u^\dagger(q,\sigma') + d v^\dagger(q,\sigma')\right)\gamma^0$$

$$(-i\vec{\gamma}\cdot\vec{\nabla} + m)\, e^{i\vec{p}\cdot\vec{x}} \left(\sum_\sigma b\, u(p,\sigma) + d^\dagger v(p,\sigma)\right)$$

$$\downarrow$$

$$+\vec{\gamma}\cdot\vec{p} + m$$

Now we chose u such that $(p^\mu\gamma_\mu - m)u = 0$ or $(p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} - m)u = 0$ for $p^0 = +\sqrt{\vec{p}^2 + m^2}$

so $(+\vec{\gamma}\cdot\vec{p} + m)\,u(p,\sigma) = \gamma^0 E u$

Similarly for v but for $p^0 = -\sqrt{\vec{p}^2 + m^2}$. And $u^\dagger_\sigma \gamma^{0\,2} u_{\sigma} = 2E$

$$v^\dagger \gamma^{0\,2} v = 2E > 0$$

~~$H = \int d^3x\, e^{i\vec{x}\cdot(p-q)}$~~

~~forces $p=q$ ...~~

And $\int d^3x\, e^{i(\vec{p}-\vec{q})\cdot x} \to (2\pi)^3 \delta^3(\vec{p}-\vec{q})$ does q-int.

$$H = \int \frac{d^3p}{(2\pi)^3 2p^0} \sum_\sigma \left[E_p\, b^\dagger b_{p,\sigma} - E_p\, d d^\dagger_{p,\sigma} \right]$$

If I really think $\{b, b^\dagger\}$, $\{d, d^\dagger\}$ obey usual commutation relations, I am <u>dead</u>.

Theorem (Spin-Statistics)

In relativistic QFT, at spacelike $(x-y)$ eg $(x-y)^\mu (x-y)_\mu < 0$

for fields φ, ψ, we have $[\varphi(x), \psi(y)] = 0$ if <u>either</u> field has integer spin

but $\{\psi(x), \psi(y)\} = 0$ if both have odd-half-integer spin.

(there are exceptions but only stupid ones. Theorem holds in 4-dim, <u>NOT</u> in some lower dimensions)

Proof: Wightman & Streater, pp 146–161

That is: $\psi(x)\psi(y) + \psi(y)\psi(x) = 0$ $\quad (x-y)$ spacelike

$\{\psi_a(x), \psi^\dagger_b(y)\} = \delta^3(x-y)\,\delta_{ab}$ $\quad$ vanishes if spacelike

$\{\psi, \psi^\dagger\} = 0$

<u>But</u> $[\psi_a, \varphi] = 0$ if φ <u>scalar</u> ψ spinor.

Swallow that. Now on - be super-careful about order of Ferm. operators, eg,

Naively $\Rightarrow$ $\psi_\alpha \psi^\alpha$ for one Weyl field $= \underbrace{\epsilon_{ab}}_{\text{antisymm}} \underbrace{\psi^b \psi^a}_{\text{symm}}$ should $= 0$

But no! $\psi^b \psi^a = -\psi^a \psi^b$. Ha! So it's not zero! $\psi^1\psi^2 - \psi^2\psi^1 = 2\psi^1\psi^2$ not $0\psi^1\psi^2$!! Ha!

I never switched order of $\psi^\dagger, \psi$ or $b^\dagger, b$ or $d, d^\dagger$ in finding H.

So $$H = \int \frac{d^3p}{(2\pi)^3 2p^0} \sum_\sigma \left(E_p\, b^\dagger_{p,\sigma} b_{p,\sigma} - E_p\, d_{p,\sigma} d^\dagger_{p,\sigma} \right)$$ is OK!

Because $d_p d_p^\dagger = -d_p^\dagger d_p + \nabla E_p$ eg, it's $\{d_p, d_k^\dagger\} = (2\pi)^3 2E_p\, \delta^3(p-k)$ not $[\ \ ]$.

Recall, the $(2\pi)^3 \delta^3(p-p)$ should be understood as vol. of space...

$$H = \int \frac{d^3p}{(2\pi)^3 2p^0} \sum_\sigma \left(E_p \left(b^\dagger_{p,\sigma} b_{p,\sigma} + d^\dagger_{p,\sigma} d_{p,\sigma} \right) - E_p^2\, \text{Vol} \right)$$

↑ another vacuum energy - this time negative.

But what is a creation operator that obeys

$\{a, a\} = 0$ and $\{a, a^\dagger\} = 1$??

The Fermionic SHO

Suppose I have an op a, its H-conj $a^\dagger$

and $\{a,a\}=0=\{a^\dagger,a^\dagger\}$ but $\{a,a^\dagger\}=1$.

$a^2=0$ Nilpotent

$a^{\dagger 2}=0$ Nilpotent

$a a^\dagger a = \{a,a^\dagger\}a - a^\dagger \overset{0}{a a}$

$= a$

and $a^\dagger a a^\dagger = a^\dagger$

So any long string of a's and $a^\dagger$'s can be reduced to:

$1, a, a^\dagger, a^\dagger a$ 4 independent operators.

H-space must be 2-component if there are only 4 op's

Assume $|\psi\rangle$ exists. If $a|\psi\rangle=0$, ~~normalize~~

then define $|0\rangle = \dfrac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}}$

If $a|\psi\rangle \neq 0$, define $|0\rangle = \dfrac{a|\psi\rangle}{\sqrt{\langle\psi|a^\dagger a|\psi\rangle}}$ so $\langle 0|0\rangle=1$, $a|0\rangle=0$

Now name $a^\dagger|0\rangle=|1\rangle$

note, $\langle 1|1\rangle = \langle 0|a a^\dagger|0\rangle = \langle 0|(1-a^\dagger a)|0\rangle$

$= \langle 0|0\rangle = 1$ properly norm'd

then $a|1\rangle = |0\rangle$ $a^\dagger|1\rangle=0$

$a|0\rangle=0$ $a^\dagger|0\rangle=|1\rangle$

$a^\dagger a|0\rangle = 0$ $a^\dagger a|1\rangle=|1\rangle$

a: lowering op

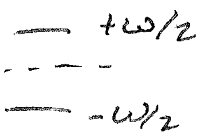
$a^\dagger$: raising op

$a^\dagger a$: number op.

Closed algebra of states.

Matrix notation, $a = \begin{bmatrix}0&0\\1&0\end{bmatrix}$ $a^\dagger \doteq \begin{bmatrix}0&1\\0&0\end{bmatrix}$ $a^\dagger a = \begin{bmatrix}1&0\\0&0\end{bmatrix}$ $\mathbb{1} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

What is this? QM of 2-state system



$H = \frac{\omega}{2}(a^\dagger a \mp a a^\dagger)$

$= \omega(a^\dagger a - \frac{1}{2})$

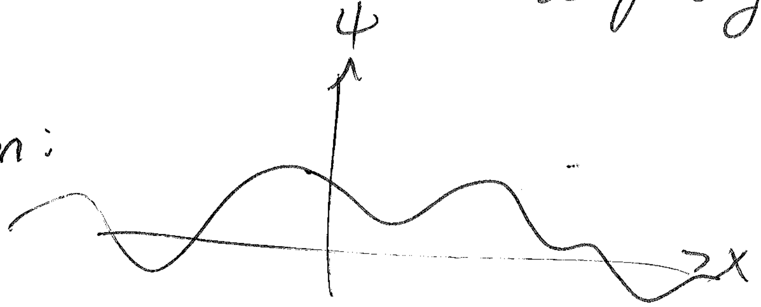
Most general state

$|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$ is $\mathbb{C}$ function over 2 points

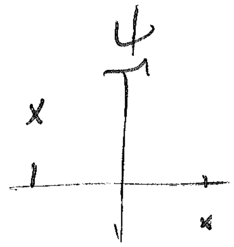
not $\mathbb{C}$ function of x $\psi(x)$ like SHO

Way way simpler.

Boson:



Fermion



classical state space → 2 points.