Fermionic Path Integral

Just the Facts

I can write a path $\int$ for a theory containing scalars:
for instance

$$\mathcal{L}(\hat\varphi, \hat\psi, \partial_\mu\hat\varphi, \partial_\mu\hat\psi) = \tfrac{1}{2}\partial_\mu\hat\varphi\,\partial^\mu\hat\varphi - V(\hat\varphi) + \hat{\bar\psi}(i\not\partial - m)\hat\psi + g\,\hat\varphi\,\hat{\bar\psi}\hat\psi$$

$$Z(J,\bar\eta,\eta) = \int \mathcal{D}(\bar\psi,\psi,\varphi)\, \exp i\int d^4x\,\left(\mathcal{L} - J\phi - \bar\eta\psi - \bar\psi\eta\right)$$

Looks just like bosonic path $\int$. But $\bar\psi, \psi, \bar\eta, \eta$ are anticommuting numbers (Grassmann's)
Order matters, and there are $-$ signs

Inside, operators $\hat\varphi, \hat{\bar\psi}, \hat\psi$ replaced with integration variables $\varphi, \bar\psi, \psi$

For instance: If I want $\langle 0|\psi\bar\psi\varphi|0\rangle$

I use

$$\frac{i\delta}{\delta J(x)} e^{i\int d^4y\,(-J(y)\varphi(y))} = \varphi(x)e^{i(\dots)}$$

$$\frac{i\delta}{\delta\bar\eta(x)} e^{i\int d^4y\,(-\bar\eta\psi(y))} = \psi(x)e^{i(\dots)}$$

$$-\frac{i\delta}{\delta\eta(x)} e^{i\int d^4y\,(-\bar\psi\eta)} = \bar\psi(x)e^{i(\dots)}$$

why $-i$ this time?

To write $\langle 0|\psi(x)\bar\psi(y)\varphi(z)|0\rangle = \dfrac{i\delta}{\delta\bar\eta(x)}\dfrac{-i\delta}{\delta\eta(y)}\dfrac{i\delta}{\delta J(z)} Z(J,\bar\eta,\eta)\Big|_{J,\bar\eta,\eta=0}$

but note $\langle 0|\bar\psi\psi\varphi|0\rangle = -\langle 0|\psi\bar\psi\varphi|0\rangle$

and so I must also have $\dfrac{\delta}{\delta\bar\eta(x)}\dfrac{\delta}{\delta\eta(y)} = -\dfrac{\delta}{\delta\eta(y)}\dfrac{\delta}{\delta\bar\eta(x)}$

I also have $\int d\bar{\psi}\, d\psi\; e^{i\int \bar{\psi}(i\partial\!\!\!/ - m)\psi}$

$= \mathrm{Det}[i(i\partial\!\!\!/ - m)]$ not Det^{-1} as you may have guessed

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Formally define $S_F(x-y)$ as inverse of $i\partial\!\!\!/ - m$, eg,

$$(i\partial\!\!\!/_x - m)\, S_F(x-y) = \delta^4(x-y)$$

Claim: momentum-space version $(p\!\!\!/ - m) S(p) = \mathbb{1}$

$$S(p) \text{ "}=\text{" } \frac{1}{p\!\!\!/ - m} \text{ means } \frac{p\!\!\!/ + m}{p^2 - m^2 + i\varepsilon}$$

because $(p\!\!\!/ - m)\dfrac{p\!\!\!/ + m}{p^2 - m^2 + i\varepsilon} = \dfrac{p\!\!\!/ p\!\!\!/ - m^2}{p^2 - m^2 + i\varepsilon} = \dfrac{p^2 - m^2}{p^2 - m^2 + i\varepsilon}$

T - boundary cond. again.

then $iS_F(x-y) = \langle 0 | T(\psi(x)\bar{\psi}(y)) | 0 \rangle_{\text{free thy}}$

check: we solved the free thy.

Use explicit form of $\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_\sigma \left(u_{\sigma p} b_{\sigma p} e^{ip\cdot x} + v_{\sigma p} d^\dagger_{\sigma p} e^{-ip\cdot x} \right)$

and anti-commutation of b, d, etc.

and $\sum_\sigma u\bar{u}(p,\sigma) = p\!\!\!/ + m$

As before, complete the square...

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; e^{\,i\int \underbrace{\bar{\psi}(S_F^{-1})\psi + \bar{\eta}\psi - \bar{\psi}\eta}_{(\bar{\psi}-\bar{\eta}S_F)S_F^{-1}(\psi - S_F\eta) - \bar{\eta}S_F\eta}}$$

$$= \operatorname{Det} S_F^{-1}\, \exp -i\int \bar{\eta}(x) S_F(x,y)\,\eta(y)\, d^4x\, d^4y$$

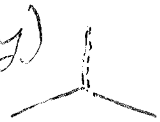
So as before, you can perform ψ-integral.

Of course one must also turn, e.g.

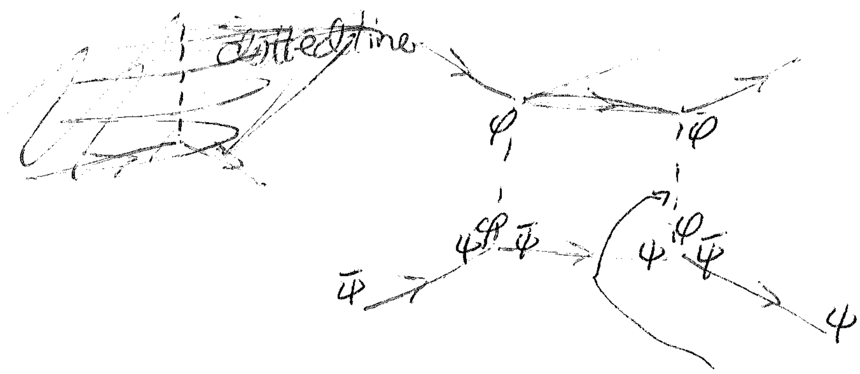
$$e^{-ig\int d^4x\, \varphi\bar{\psi}\psi} = e^{-ig\int d^4x \frac{i\partial}{\partial J(x)}\frac{\partial}{\partial \eta(x)}\frac{\partial}{\partial \bar{\eta}(x)}}$$

pull out of integral

Gives rise to vertices $(-ig)$

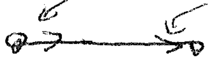


To distinguish scalars from spinors, use convention



dotted line indicates scalar propagator $\Delta(q)$ or $\frac{i\delta}{\delta J}$ at a vertex

Line with arrow for spinor propagator $S_F(p)$ or "tab" with arrow. Arrow goes from $\bar{\psi}$ or $\frac{\delta}{\delta\eta}$ to ψ or $\frac{\delta}{\delta\bar{\eta}}$



Ok, but what does ψ or $\bar{\psi}$ operator have to do with making / destroying particle?
what n-point function should I evaluate to find, say, <u>scattering</u> $\psi\psi \to \psi\psi$?

Think back to scalar case. Imagine I had

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_a M_{ab} \partial^\mu \varphi_b - V(\varphi)$$

M_{ab} invertible real symmetric <u>matrix</u>

Then $\langle 0|T(\varphi_a(x)\, \varphi_b(y))|0\rangle = i M^{-1}_{ab} \Delta_F(x-y)$

$$p\text{-space}: \int d^4x\, e^{ipx} \,(\ldots) = \frac{i M^{-1}_{ab}}{p^2 - m^2 + i\epsilon}$$

φ_a does <u>not</u> create single particle w. usual normalization.
spectral function ρ_{ab} is non-diagonal $M^{-1}_{ab}\, 2\pi\delta(p^2-m^2)$.

What do I do? Solve eigenvalue / eigenvector problem for M_{ab}

$$M_{ab}\, \xi_{b,i} = \lambda_i\, \xi_{a,i}$$

i: index of eigenvector
λ_i: eigenvalue (positive real)

$$M_{ab} = \sum_i \lambda_i \underbrace{\xi_{ai}\xi_{bi}}_{\text{matrix}}$$

true if ξ_{ai} chosen unit-normal
Automatically orthogonal!!

define $\tilde{\varphi}_i = \sum_a \lambda_i^{1/2} \xi_{ai} \varphi_a$

$$\langle 0|T(\tilde{\varphi}_i \tilde{\varphi}_j)|0\rangle = \lambda_i^{1/2}\lambda_j^{1/2} \underbrace{\xi_{ai}\xi_{bj}\, i M^{-1}_{ab}}_{\lambda_i^{-1}\delta_{ij}} \Delta_F(x-y) = \delta_{ij}\, i \Delta_F(x-y)$$

<u>These</u> act like normal-old particle creators.

Continuing this example

Scatter 4 particles: I want to evaluate

$\langle 0|T(\tilde{\varphi}_i \tilde{\varphi}_j \tilde{\varphi}_k \tilde{\varphi}_l)|0\rangle$

$$= \sum_{abcd} \lambda_i^{1/2} \lambda_j^{1/2} \lambda_k^{1/2} \lambda_l^{1/2} \langle 0|T(\varphi_a \varphi_b \varphi_c \varphi_d)|0\rangle \xi_{ai} \xi_{bj} \xi_{ck} \xi_{dl}$$

$\parallel$

 $= \cancel{A}\, G(p_1) G(p_2) G(p_3) G(p_4)\, i\delta^4 \mathcal{M}$

$$= \sum_{abcd} \lambda_i^{1/2} \lambda_j^{1/2} \lambda_k^{1/2} \lambda_l^{1/2} \xi_{ai} \xi_{bj} \xi_{ck} \xi_{dl} \frac{i^4 M^{-1}_{aa'} M^{-1}_{bb'} M^{-1}_{cc'} M^{-1}_{dd'}}{(p_1^2 - m^2 + i\epsilon)(p_2^2 - m^2 + i\epsilon)(k_1^2 - m^2 + i\epsilon)(k_2^2 - m^2 + i\epsilon)} \times \delta^4(\Sigma)$$

$\times (\lambda$ or whatever$)$

$$\lambda_i^{1/2} \xi_{ai} M^{-1}_{aa'} = \lambda_i^{-1/2} \xi_{ai}$$

$= $ eigenthingy of M^{-1}, eg. $M^{-1}_{ab} = \sum_i (\lambda_i^{-1/2} \xi_{ai})(\xi_{bi} \lambda_i^{-1/2})$

$$= \sum_{abcd} \underbrace{\lambda_i^{-1/2} \xi_{ai}}\, \lambda_j^{-1/2} \xi_{bj}\, \lambda_k^{-1/2} \xi_{ck}\, \lambda_d^{-1/2} \xi_{dl} \times (\lambda_{abcd} \text{ or whatever})$$

these are needed to get me from

- basis of things which correctly make 1 particles well normalized, to
- fields φ_a which I actually have & which I can find correlation functions for.

For me with spinors, I know $S_F = \dfrac{(\not{p} + m)_{ab}}{p^2 - m^2 + i\epsilon}$ " $=$ " $\dfrac{M^{-1}_{ab}}{p^2 - m^2 + i\epsilon}$

My $\lambda_i^{-1/2} \xi_{ai}$ are the "roots" of $\not{p} + m$, ~~eg. things where etc~~

Call $i \to \sigma$

a = spinor index usually suppressed.

$$M^{-1} = \sum_i \lambda_i^{-1/2} \xi^*_{ai} \lambda^{-1/2} \xi^T_{bi} \Rightarrow \not{p}+m = \sum_\sigma u_{\sigma p} \bar{u}_{\sigma p}$$

oh hey - that's exactly what my $u_{\sigma p}$ do! So $u_{\sigma p}$ is $\equiv$ my $\lambda^{-1/2}\xi$ object. I picked that weird $\sqrt{2E} \times \left[\text{orthonormal spinor}\right]$ normalization to match eigenval. spectrum of $\not{p}+m$.

What if its an antiparticle - if $p^0<0$ so it's $-\not{p}+m$?

~~Conclusion~~ Then use v. And note funny sign.

Rules: Don't compute, say, $\langle 0|T(\psi\psi\bar{\psi}\bar{\psi})|0\rangle$ to find $\psi\psi \to \psi\psi$ scattering.

Compute external propagator $\left(\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\right)$ amputated correlator, and then pin on

$u_{\sigma p}$ for incoming ψ ($\bar{\psi}\cdot op$)

$\bar{u}_{\sigma p}$ for outgoing ψ ($\psi\cdot op$)

$V_{\sigma p}$ for outgoing $\bar{\psi}$ ($\bar{\psi}\cdot op$)

$\bar{V}_{\sigma p}$ for incoming $\bar{\psi}$ ($\psi\cdot op$)