Example of Relativistic classical F.T.

Suppose each pt. in space x has a number $\varphi(x)$ associated.
that's a scalar field. φ varies in sp. and time $x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$

Invariant distance $t^2 - \vec{x}^2 = \underbrace{[t \; -x \; -y \; -z]}_{x_\mu} \underbrace{\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}}_{= x^\mu}$

$x_\mu = g_{\mu\nu} x^\nu$

"metric" $g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$ (blank entries mean 0)

and the $\sum_\nu$ implied. Everytime index appears 2x, once ^{up} once $_{down}$, sum it.
If index appears 2 times ^{up} or 2 $_{down}$ or more than 2x, You Screwed Up.
Follow these rules and everything will always be relativistically invariant.

Write $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$ the derivative covariant 4-vector.

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What could $\mathcal{L}(\varphi, \partial_\mu \varphi)$ look like? $S = \int d^4x \, \mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$

Rules: must be spacetime scalar
Since I want $\int d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi) = S$, dont worry about total deriv's.

Use dimension counting to help narrow possibilities.

$\mathcal{L}$ terms with no deriv's:

$$\mathcal{L} = c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4 + \ldots$$

we'll see.

c_0: yes. But boring

$c_2\varphi^2$: yes

$c_4\varphi^4$: yes

$c_1\varphi$: yes. But I can eliminate with shift $\varphi \to \varphi + \text{const}$.

$c_3\varphi^3$: yes but I will ignore it today.
For instance maybe $\mathcal{L}$ unchg'd on $\varphi \to -\varphi$

Terms w. 2 deriv's:

$\partial_\mu \partial^\mu \varphi$ (or $g^{\mu\nu}\partial_\mu\partial_\nu\varphi$) no, it's a total deriv.

$$\int d^4x\, \partial_\mu[\partial^\mu\varphi] = \int \partial^\mu\varphi \cdot d\Sigma_\mu$$

I don't know if space has a boundary. Can't matter.

Boundary

$(\partial_\mu\varphi)(\partial^\mu\varphi)$, $\varphi\partial_\mu\partial^\mu\varphi$: yes but only one of them.

$\int d^4x\, \partial_\mu(\varphi\partial^\mu\varphi)$ is total deriv.

It's also $\int d^4x\left[(\partial_\mu\varphi)(\partial^\mu\varphi) + \varphi\partial_\mu\partial^\mu\varphi\right]$

So I can absorb $\varphi\partial_\mu\partial^\mu\varphi$ by adding total deriv & turning it into $-\partial_\mu\varphi\partial^\mu\varphi$.

$\varphi\partial_\mu\varphi\partial^\mu\varphi$ yes but see below

$\partial_\mu\varphi$ or $\varphi\partial_\mu\varphi$ or... : $\underline{\underline{\text{no}}}$. Not Lorentz inv. as this is a 4-<u>vector</u> (uncontracted index) not a scalar.

So think about

$$S = \int d^4x \left[K_1 \partial_\mu \varphi \partial^\mu \varphi + C_0 + C_2 \varphi^2 + C_4 \varphi^4 + \left(C_6 \varphi^6 \text{ or } \varphi^2 \partial_\mu \varphi \partial^\mu \varphi \right) \right]$$

choose to normalize φ : $\varphi_{new} = \sqrt{\frac{2}{K_1}}\, \varphi_{old}$

to make this term $\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi$

S: units of action $\underset{\mathcal{E}}{\underline{\mathcal{E}}} \cdot t$ (same as $\hbar$. If you do QM, $\hbar \to 1$ units are nice, and S is dim'less)

$\int d^4x$: units of t^4 (units on x, t are same. We use seconds and light-seconds)

∂_μ : t^{-1} a deriv undoes a length: $\partial_\mu = \frac{\partial}{\partial x^\mu}$ ← length in denom.

so $2[\varphi] = [S] - 4[dx] - 2[\partial_\mu]$ $\quad$ $[\;]$: units of

$= \mathcal{E} \cdot t \cdot t^{-4} \cdot t^{+2} = \mathcal{E}/t$ energy/time

$[\varphi] = \sqrt{\mathcal{E}/t}$.

$[C_0] = \mathcal{E}/t^3$ energy density of empty space.
Irrelevant unless you are doing G.R.

$[C_2] = 1/t^2$ frequency^2. Name $C_2 = \frac{\omega_0^2}{2}$ (or $\frac{m^2 c^4}{2}$)

$[C_4] = 1/t\mathcal{E}$ same as $\hbar^{-1}$. Huh. Name it $\lambda/\hbar \cdot 4!$ Assume $\lambda \sim 1$

$[C_6] = \frac{1}{\mathcal{E}^2} = \frac{t^2}{\mathcal{E}^2 t^2} = \frac{t^2}{\hbar^2}$. Assume of order $\frac{(\text{Planck length})^2}{\hbar^2}$ really small. Neglect!?

Euler-Lagrange eq?

$$\frac{\delta S}{\delta \varphi(x)} = 0 = \frac{\delta}{\delta \varphi(x)} \int d^4y \left(\frac{1}{2} \partial_\mu \varphi(y) \partial^\mu \varphi(y) \mp \frac{\omega^2}{2} \varphi^2_{(y)} \mp \frac{\lambda}{4!\hbar} \varphi^4_{(y)} \right)$$

$$\frac{\delta \varphi(y)}{\delta \varphi(x)} = \delta^4(x-y)$$

$$\frac{\delta\, \partial_\mu \varphi(y)}{\delta\, \varphi(x)} = \partial_\mu \delta^4(x-y)$$ requires $\int$ by parts to interpret.

$$\frac{\delta}{\delta \varphi(x)} \int d^4y\ \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi(y) = \int d^4y\ \frac{1}{2} \Big[\underbrace{\partial^\mu \varphi\, \partial_\mu \delta^4(x-y)}_{-(\partial^\mu \partial_\mu \varphi)\delta^4(x-y)} + \underbrace{\partial_\mu \varphi\, \partial^\mu \delta^4(x-y)}_{-(\partial_\mu \partial^\mu \varphi)\delta^4(x-y)} \Big]$$

$$= -\partial^\mu \partial_\mu \varphi(x)$$

no boundary terms as $\delta^4(x-y) = 0$ there!!

$$\frac{\delta}{\delta \varphi(x)} \int d^4y\ \frac{\omega^2}{2} \varphi^2(y) = \omega^2 \varphi(x).$$

EOM: $\partial_\mu \partial^\mu \varphi(x) = -\omega^2 \varphi(x) \mp \frac{\lambda}{3!\hbar} \varphi^3(x)$ Euler-Lagrange Eq. "Klein-Gorden Eq."

This is general pattern. I can name $\pi^\mu = \frac{\delta \mathcal{L}}{\partial(\partial_\mu \varphi)}$ Canonical 4-m.. ~~..~~ density

then $\partial_\mu \pi^\mu = \frac{\delta \mathcal{L}}{\delta \varphi}$

Hamiltonian?

$$L = \int d^3x \left[\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right]$$

$$= \int d^3x \Big(\underbrace{\tfrac{1}{2}\partial_0\varphi\partial_0\varphi}_{\text{Kin}} \underbrace{- \tfrac{1}{2}\nabla\phi\cdot\nabla\phi - V(\phi)}_{\text{"Pot"}}\Big)$$

$\pi(x) \equiv \dfrac{\partial L}{\partial(\dot\varphi(x))} = \dot\varphi(x)$ technically there is a Kroneker vs Dirac δ issue here!!

$$H = \sum P\dot{Q} - L \longrightarrow \int d^3x\left(\pi\dot\varphi - \left(\tfrac{1}{2}\partial_0\varphi\partial_0\varphi - \tfrac{1}{2}\nabla\varphi\cdot\nabla\varphi - V\right)\right)$$

$$H = \int d^3x\Big(\underbrace{\tfrac{1}{2}\dot\varphi^2}_{\text{positive}} + \underbrace{\tfrac{1}{2}\nabla\phi\cdot\nabla\phi}_{\text{positive}} + V(\phi)\Big)$$

bounded from below if V is.

$V(\phi)$ must be bounded from below

$$V(\phi) = C_0 + \frac{\omega^2}{2}\phi^2 + \frac{g}{6}\phi^3 + \underbrace{\frac{\lambda}{24}}_{\lambda \geq 0 \text{ required}}\phi^4$$

$\omega^2 > 0$ if $g, \lambda = 0$ $\quad g = 0$ if $\lambda = 0$

Otherwise, Need Not!! (??)

H is not Lorentz invariant. It is the thing which must be bounded from below.

Soving

$$\partial_\mu \partial^\mu \varphi = -\frac{\partial V}{\partial \varphi} \qquad \text{for } \frac{\omega_0^2}{2}\varphi^2 + \frac{\lambda}{24}\varphi^4, \ldots$$

$$\partial_0^2 \varphi_{(x)} = \nabla^2 \varphi_{(x)} - \omega_0^2 \varphi_{(x)} - \frac{\lambda}{6}\varphi^3_{(x)} \qquad \text{nonlinear PDE}$$

no general solution techniques exist. But we do have
- lattice: solve discrete-space version numerically
- pert. theory: treat $\lambda\varphi^3$ as "small" and "expand"

Lucky case: if $\lambda = 0$

$$\partial_0^2 \varphi = \nabla^2 \varphi - \omega_0^2 \varphi \qquad \text{almost the wave equation}$$

Solve by Fourier

$$\phi(x,t) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p},t)$$

$$\phi(\vec{p},t) = \int d^3x \, e^{-i\vec{p}\cdot\vec{x}} \phi(x,t)$$

Insert into EOM, apply $\int d^3x \, e^{-i\vec{p}\cdot\vec{x}}$, and find

$$\partial_0^2 \varphi(p,t) = (-p^2 - \omega_0^2)\varphi(p,t)$$

$$\varphi(p,t) = A_p \cos(\omega t + \phi_p) \qquad \omega = \sqrt{p^2 + \omega_0^2}$$

Find A_p ϕ_p from initial conditions Done

Also, Green's function techniques can be applied.

More elaborate example: N scalars $\varphi_1 \dots \varphi_N$ collect. φ_a

Require invariance: $\mathcal{L}$ same under $\varphi_a \to \mathcal{O}_{ab}\varphi_b$. $\mathcal{O}_{ab}$ orthogonal matrix $\mathcal{O}^{-1}_{ab} = \mathcal{O}^{T}_{ab}$ (note $\mathcal{O}_{ab}$ all real)

Sum all repeated a-type indices so $\partial_\mu\varphi_a\partial^\mu\varphi_a \equiv \sum_{\mu,a}\partial_\mu\varphi_a\partial^\mu\varphi_a$

Most general invariant $\mathcal{L}$ is

$$S = \int d^4x\left[\frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a \mp \frac{\omega_0^2}{2}\varphi_a\varphi_a \mp \frac{\lambda}{8h}(\varphi_a\varphi_a)(\varphi_b\varphi_b)\right]$$

E-L eq: $\partial_\mu\partial^\mu\varphi_a = -\omega_0^2\varphi_a \mp \frac{\lambda}{2h}\varphi_a\varphi_b\varphi_b$

Something fancy is happening here: there is a conserved current!

For every pair $a \neq b$ ($\frac{N(N-1)}{2}$ pairs), the quantity

$J^\mu_{ab} = [\varphi_a\partial^\mu\varphi_b - \varphi_b\partial^\mu\varphi_a]$ is a conserved current, in sense

$\partial_\mu J^\mu_{ab} = 0$. Let's see that.

$$\partial_\mu J^\mu_{ab} = \underbrace{\cancel{\partial_\mu\varphi_a\partial^\mu\varphi_b}}_{= g_{\mu\nu}\partial^\nu\varphi_a\partial^\mu\varphi_b} - \cancel{\partial_\mu\varphi_b\partial^\mu\varphi_a} + \varphi_a\partial_\mu\partial^\mu\varphi_b - \varphi_b\partial^\mu\partial_\mu\varphi_a$$

Rename $\mu \leftrightarrow \nu$, use summ $g_{\mu\nu} = g_{\nu\mu}$: $g_{\mu\nu}\partial^\mu\varphi_a\partial^\nu\varphi_b = \partial^\mu\varphi_a\partial_\mu\varphi_b$ cancels...

Now use EOM:

make sure to use indep. dummy variable!

$$\partial_\mu J^\mu_{ab} = \varphi_a\partial_\mu\partial^\mu\varphi_b - \varphi_b\partial^\mu\partial_\mu\varphi_a$$

$$= \varphi_a\frac{\omega^2}{2}\varphi_b + \frac{\lambda}{2h}\varphi_a\varphi_b\varphi_c\varphi_c - \varphi_b\frac{\omega^2}{2}\varphi_a \mp \frac{\lambda}{2h}\varphi_b\varphi_a\varphi_c\varphi_c$$

$\partial_\mu J^\mu_{ab} = 0$

Oh-they cancel. But how did we know that would happen??