Signs and fermions

Order of operators matters in fermionic system.

But sign of external state also not always uniquely defined.

eg, if in-state has two fermions $\psi_1, \psi_2$ is it

$$|\psi_1(p_1)\,\psi_2(p_2)\rangle = \int \bar{\psi}_1(x_1)\bar{\psi}_2(x_2)|0\rangle \; e^{i(p_1\cdot x_1 + p_2\cdot x_2)} d^4x_1 d^4x_2$$

$$\overset{\text{or}}{=} \int \bar{\psi}_2(x_2)\,\bar{\psi}_1(x_1)|0\rangle \qquad ?$$

I have to choose one by convention.

Overall sign in $\mathcal{M}$.

Then when I take $\mathcal{M}^*\mathcal{M}$ to get rate $\rightarrow$ doesn't matter!!

But relative signs can matter. Here is where:

If ~~as~~ two processes connect fermions together differently

If there is an "internal loop" of fermions

Illustration: Yukawa Theory $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \bar{\psi}(i\partial\!\!\!/ - m)\psi$
$- y\varphi\bar{\psi}\psi - \frac{\lambda}{24}\varphi^4 - \frac{M^2}{2}\varphi^2$

Consider 3 processes

$\varphi\psi \rightarrow \varphi\psi$ scatt (for practice)

$\psi\psi \rightarrow \psi\psi$ scatt (exchange-signs)

$\varphi\varphi \rightarrow \varphi\varphi$ if $y^4 \sim \lambda$ (you'll see why!)

# $\varphi\psi \to \varphi\psi$ Feynman rules

$$i\mathcal{L} = i\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{i m^2\varphi^2}{2} + i\bar\psi(i\partial\!\!\!/ - m)\psi - iy\varphi\bar\psi\psi - \frac{i\lambda}{24}\varphi^4$$

$$\frac{i}{p^2-m^2+i\varepsilon}$$

inverse of thing between $\varphi$'s

$\varphi(-i\partial_\mu\partial^\mu - m^2)\varphi$

$= \varphi(i(p^2-m^2))\varphi$

$$\frac{i(p\!\!\!/+m)}{p^2-m^2+i\varepsilon} = \text{"}\frac{i}{p\!\!\!/-m}\text{"}$$

inverse of thing between $\bar\psi$ and $\psi$

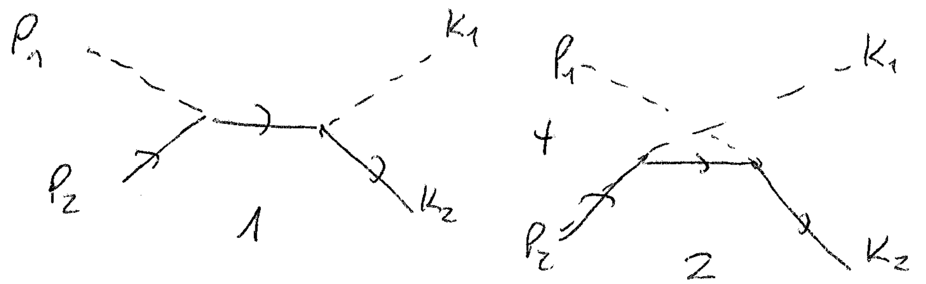
$-iy$ 

$-\frac{i\lambda}{24}$ X

note symm factor

And overall $i$ from sign convention of $\mathcal{M}$.

$\delta^4(\Sigma p)\mathcal{M} = -i(G's)\langle\varphi\varphi\ldots\rangle\ldots$



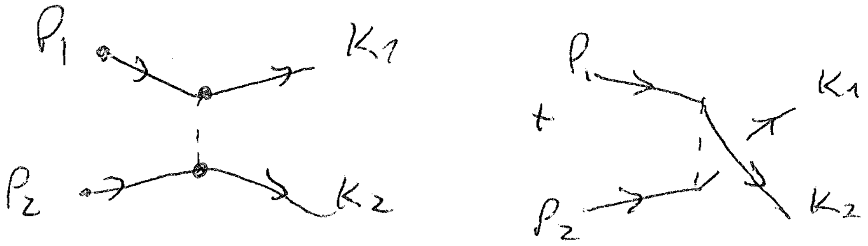

$$\mathcal{M}_1 = \bar u(\sigma_2,k_2)(-iy)\frac{i(p\!\!\!/_1+p\!\!\!/_2+m)}{(p_1+p_2)^2-m^2}(-iy)u(\sigma_1 p_2) = -iy^2\frac{\bar u(\sigma_2 k_2)(p\!\!\!/_1+p\!\!\!/_2+m)u(\sigma_1 p_2)}{s-m^2}$$

$$\mathcal{M}_2 = \bar u(\sigma_2 k_2)(-iy)\frac{i(p\!\!\!/_2-k\!\!\!/_1+m)}{(p_2-k_1)^2-m^2}(-iy)u(\sigma_1 p_2) = -iy^2\bar u(\sigma_2 k_2)\frac{(p\!\!\!/_2-k\!\!\!/_1+m)}{u-m^2}u(\sigma_1 p_2)$$

Relative sign is same - as they happen in same fermionic order

evaluation - trace methods (using it)

# $\psi\psi \to \psi\psi$

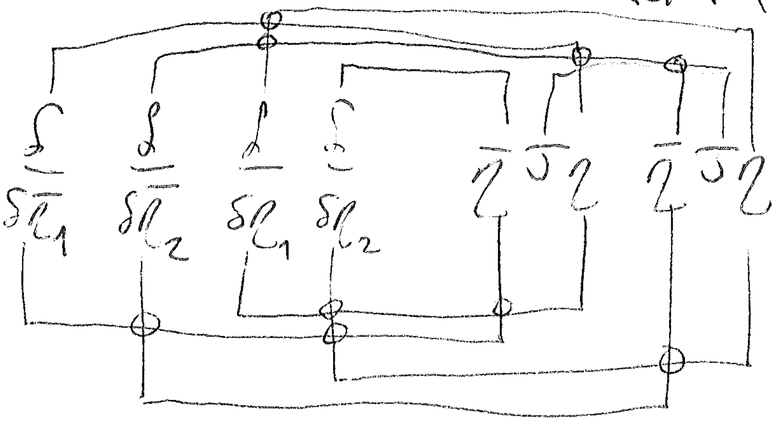



Beware!

Suppose I choose <u>convention</u>, in state $= b^{+}_{p_2} b^{+}_{p_1} |0\rangle$

out state $= \langle 0 | b_{k_1} b_{k_2}$

Then I <u>want</u> (up to $u$'s) $\langle 0 | T\left( \psi(k_1)\psi(k_2)\bar{\psi}(p_2)\bar{\psi}(p_1) \right)|0\rangle$




4 order changes for diagram 2

5 order changes for diagram 1

<u>Different signs!</u> so $\mathcal{M}_1 + \mathcal{M}_2 = (iy)^2 \dfrac{\bar{u}(k_1)u(p_1)\,\bar{u}(k_2)u(p_2)\, i}{(p_1-k_1)^2 - M^2 + i\varepsilon}$

$- (iy)^2\, \dfrac{i\, \bar{u}(k_1)u(p_2)\,\bar{u}(k_2)u(p_1)}{(p_1-k_2)^2 - M^2 + i\varepsilon}$

relative — sign is physical.

Arises when assignment of incoming fermion lines to outgoing involves an <u>odd</u> permutation

$\begin{pmatrix}1\\2\end{pmatrix} \to \begin{pmatrix}1\\2\end{pmatrix}$ vs $\underbrace{\begin{pmatrix}1\\2\end{pmatrix} \to \begin{pmatrix}2\\1\end{pmatrix}}_{\text{odd permutation}}$

evaluation: $\bar{\mathcal{M}}_1^2 = \mathrm{Tr}\, u_{k_1}\bar{u}_{k_1} u_{p_1}\bar{u}_{p_1}\ \mathrm{Tr}\, u_{k_2}\bar{u}_{k_2} u_{p_2}\bar{u}_{p_2} = \mathrm{Tr}\,(\not{k}_1 - m)(\not{p}_1 - m)\,\mathrm{Tr}(\not{k}_2 - m)(\not{p}_2 - m)$

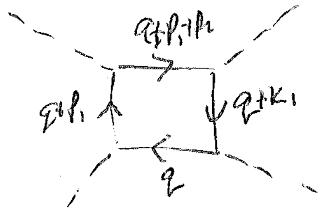
$= 16(k_1\cdot p_1 + m^2)(k_2\cdot p_2 + m^2)$

$p_1^2 - 2k_1\cdot p_1 + k_1^2 = (p_1 - k_1)^2 = t$

$2m^2 - 2k_1\cdot p_1 = t$. $\quad k_1\cdot p_1 = \frac{-t}{2} + m^2$

$\to y^4 \left( \dfrac{4(t-2m^2)^2}{(t-M^2)^2} \right) \ldots$

while $\varphi\varphi \to \varphi\varphi$ at $\mathcal{O}(y^4)$




+ 5 permutations

$$\frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_4}(\mathcal{L}_y)^4 \quad \frac{\delta}{\delta\eta}\frac{\delta}{\delta J}\frac{\delta}{\delta\bar\eta}\;\frac{\delta}{\delta\eta}\frac{\delta}{\delta J}\frac{\delta}{\delta\bar\eta}\;\frac{\delta}{\delta\eta}\frac{\delta}{\delta J}\frac{\delta}{\delta\bar\eta}\;\frac{\delta}{\delta\eta}\frac{\delta}{\delta J}\frac{\delta}{\delta\bar\eta}$$

pair them off: $\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta}e^{+i\bar\eta S\eta} = iS$

but one is $\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta} = -\frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta\eta}$

but $\frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta\eta}e^{i\bar\eta S\eta} = -iS$ sign!

Always a $-$ sign.

One $-$ sign per fermion loop because of this.

$$i\int\frac{d^4q}{(2\pi)^4}(-iy)^4\, i^4\,(-)\,\mathrm{Tr}\left(\frac{(\not q+m)(\not q+\not p_1+m)(\not q+\not p_1+\not p_2+m)(\not q+\not k_1+m)}{(q^2-m^2)((p_1+q)^2-m^2)((q+p_1+p_2)^2-m^2)((q+k_1)^2-m^2)}\right)$$

as there is a loop

oh - and this looks like, at large $q$, it will

$\to -iy^4\int d^4q\,\frac{4\cancel{q}\cancel{q}\cancel{q}\cancel{q}}{q^2q^2q^2q^2} \sim \int\frac{d^4q}{q^4}$ diverges logarithmically for large $q$? Is that a problem?

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Takeaway: $-$ signs when:

$(-)^{\#\text{ of fermion loops}}$

relative $-$ between $\mathcal{M}$-contrib's with odd permutation of in $\leftrightarrow$ out fermion lines.

That's it. Memorize & stop thinking ☺

L20 P5

$\gamma^5$ aside  note $\gamma^0\gamma^1 = \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \to K_1$

$\gamma^2\gamma^3 = \begin{bmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{bmatrix} \to J_1$

$\gamma^0\gamma^1\gamma^2\gamma^3 = +i\begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}$

$i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -\mathbb{1} & 0 \\ 0 & +\mathbb{1} \end{bmatrix}$

opposite sign for R (−) as for L(+)
and that's invariant as $K_1 J_1 = \pm i \mathbb{1}$ depending on R, L for any Weyl.

More Lorentz way to write

$$\gamma^5 = \frac{i}{24}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta \quad (\epsilon_{0123} = 1)$$

$\frac{\mathbb{1} - \gamma_5}{2}$ projects out R-comp.

$\gamma^5 = \gamma_5 = -\gamma_5$ in diff't peoples' notation

$\frac{\mathbb{1} + \gamma_5}{2}$ projects out L-comp