Continuous Symmetries again

Consider ψ ~~and~~ or a complex scalar $\Phi \equiv \frac{\varphi_r + i\varphi_i}{\sqrt{2}}$

$$\partial_\mu \Phi^* \partial^\mu \Phi = \frac{1}{2}(\partial_\mu \varphi_r \partial^\mu \varphi_r + \partial_\mu \varphi_i \partial^\mu \varphi_i)$$

$$\Phi^* \Phi = \frac{1}{2}(\varphi_r^2 + \varphi_i^2)$$

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \quad \text{or} \quad \partial_\mu \Phi^* \partial^\mu \Phi - M^2 \Phi^* \Phi - \frac{\lambda}{2}(\Phi^* \Phi)^2$$

<u>have symmetry</u>

$$\psi \to e^{i\theta}\psi \quad (\psi \to \psi + i\theta\psi) \qquad \Phi \to e^{i\theta}\Phi$$

$$\bar{\psi} \to e^{-i\theta}\bar{\psi} \quad (\bar{\psi} \to \bar{\psi} - i\theta\bar{\psi}) \qquad \Phi^* \to e^{-i\theta}\Phi^*$$

more generally could have $\psi_a \to \psi_a + i\theta_A T^A_{ab}\psi_b \equiv \psi'$ $\quad T^A_{ab}$ some Hermitian matrices

($T = 1$ for our minimal case)

Consider $\mathcal{L}$ with $\mathcal{L}(\psi') = \mathcal{L}(\psi)$ as in (restricted) Nöther case.

<u>Classically</u> we saw $J^\mu_A \equiv i T^A_{ab}\, \varphi_a \pi^\mu_b$ ~~$\pi^\mu_a \varphi_b$~~ was conserved current.

$$\pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = i\bar{\psi}\gamma^\mu \text{ for fermion, } \partial^\mu \Phi^* \text{ for scalar}$$

Careful, also need π^μ for $\bar{\psi}$ or Φ^*: 0 and $\partial^\mu \Phi$ respectively.

$$J^\mu = \bar{\psi}\gamma^\mu\psi \quad \text{or} \quad J^\mu = i(\partial^\mu \Phi^* \Phi - \Phi^* \partial^\mu \Phi)$$

In what sense is $\partial_\mu J^\mu = 0$ now that $\psi, \bar{\psi}, \Phi, \Phi^*$ are operators?

Path Integral!

$$Z = \int \mathcal{D}\Phi^*,\Phi \; \exp i\int d^4x \, \mathcal{L}(\Phi, \partial_\mu \Phi)$$

or $\mathcal{D}\bar{\psi}\psi$

change variables $\varphi_a \to \varphi_a + \theta_A^{(x)} i T^A_{ab} \varphi_b$ in general case
Now $\theta = \theta(x)$!!

Assuming $\mathcal{D}\varphi_a$ unchanged (seems safe! But!)

only change to $\mathcal{L}$ could matter.

$$\delta \int d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi) = \int d^4x \; i T^A_{ab} \theta_A(x) \varphi_b \frac{\partial \mathcal{L}}{\delta \varphi_a} - i T^A_{ab} \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} \partial_\mu(\theta_A \varphi_b)$$

after $\int$ by parts as usual

$$\partial_\mu(\theta_A \varphi_b) = \theta_A \partial_\mu \varphi_b + \varphi_b \partial_\mu \theta_A$$ allowed since I made θ_A depend on x.

The statement "$\mathcal{L}$ is unchanged" is statement

$$\theta_A i T^A_{ab} \left(\varphi_b \frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \varphi_b \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} \right) = 0 .$$ So only

$-i T^A_{ab} \pi^\mu_b \varphi_a \partial_\mu \theta_A$ remains. Expand $e^{i(S+\delta S)} \simeq (1 + i\delta S) e^{iS}$

After chg of variables $Z = \int \mathcal{D}\varphi_a \, e^{iS} \left(1 + \partial_\mu \theta_A T^A_{ab} \pi^\mu_b \varphi_a\right)$

But it was just a change of variables! Z should not change!

$$0 = \int \mathcal{D}\varphi_a \, e^{iS} \partial_\mu \theta_A \underbrace{T_{ab}^A \pi^\mu_b \varphi_a}_{-iJ^\mu_A} \quad \text{for any } \theta_A(x)$$

Is true iff $\int \mathcal{D}\varphi_a \, \partial_\mu J^\mu_A e^{iS} = 0$

so $\langle \partial_\mu J^\mu \rangle = 0$

What about, say, $\partial_\mu^{x_3} \langle 0 | T(\varphi_a(x_1)\, \varphi_b(x_2)\, J^\mu(x_3)) | 0 \rangle$?

That's question, change integration measure in

$$\frac{\partial}{\partial J_1}\frac{\partial}{\partial J_2} Z = \int \mathcal{D}\varphi_a \quad \varphi_a(x_1)\, \varphi_b(x_2)\, e^{i\int d^4x\, \mathcal{L} \cdots}$$

Now when I change variables $\varphi_a \to \varphi_a + \theta_A(x)\, i T^A_{ab} \varphi_b$

The φ's here also change.

change in $\mathcal{L}$

$$0 = (\text{chg of variables}) - (\text{original } Z)$$

$$= \int \mathcal{D}\varphi_a\, e^{iS} \Big\{ \Big[\int d^4y\; i\partial_\mu \theta_A(y)\, J^\mu_A(y)\; \varphi_a(x_1)\varphi_b(x_2) \Big]$$

$$+ i\theta_A(x_1)\, T^A_{ac} \varphi_c(x_1)\; \varphi_b(x_2)$$

$$+ i\,\theta_A(x_2)\, T^A_{bc}\, \varphi_c(x_2)\; \varphi_a(x_1) \Big\}$$

$$\langle 0 | T(\partial_\mu J^\mu_A(x_3)\, \varphi_a(x_1)\, \varphi_b(x_2)) \rangle = -\delta^4(x_3 - x_1)\, i T^A_{ac} \langle \varphi_c(x_1) \varphi_b(x_2) \rangle$$

$$- \delta^4(x_3 - x_2)\, i T^A_{bc} \langle \varphi_a(x_1) \varphi_c(x_2) \rangle$$

contact terms

Contact terms can be thought of as $\partial_\mu^{x_3}$ acting on the $\Theta(x_3^0 - x_1^0)$
hidden inside the T-ordering
rather than acting on the J^μ_A.

Go back to symmetry

Nothing changes if I replace $\psi \to e^{i\theta}\psi$

why not replace $\psi \to e^{i\theta(x)}\psi$?

Well that does change something:

$$\mathcal{L} = \bar\psi(i\not\partial - m)\psi \to \bar\psi e^{-i\theta}(i\not\partial - m)e^{i\theta}\psi$$

$$= \bar\psi \underbrace{e^{-i\theta}e^{i\theta}}_{\text{cancel}}(i\not\partial - m)\psi$$

$$+ \bar\psi e^{-i\theta}e^{i\theta}(i\gamma^\mu\partial_\mu i\theta)\,\psi$$

$\delta\mathcal{L} = -\partial_\mu\theta \underbrace{\bar\psi\gamma^\mu\psi}_{J^\mu}$ does change. And it involves that J^μ. Dang.

But I can make it not matter as follows.

- Add vector field A^μ
- Replace $\partial_\mu\psi \to (\partial_\mu - iA_\mu)\underset{\Phi}{\psi}$ $\left[\ \partial_\mu\bar\psi \to (\partial_\mu + iA_\mu)\underset{\Phi^*}{\bar\psi}\ \right]$

or more generally $\partial_\mu\varphi_a \to (\partial_\mu\delta_{ab} - iA^A_\mu T^A_{ab})\varphi_b \equiv D^{ab}_\mu\varphi_b$

$D_\mu\psi = (\partial_\mu - iA_\mu)\psi$ called "covariant derivative" (sorry)

Also allow A_μ to change, $A_\mu \to A_\mu + \delta A_\mu$

$$\mathcal{L} = \bar\psi(i\not\partial + \not A - m)\psi \to \bar\psi e^{-i\theta}(i\not\partial + \not A + \delta\not A - m)e^{i\theta}\psi$$

$$= \bar\psi(i\not\partial + \not A - m - (\partial_\mu\theta)\gamma^\mu + \delta A_\mu\gamma^\mu)\psi$$

Just choose $\delta A_\mu = \partial_\mu\theta$ and these will cancel!!

Just a formal trick?

Make A_μ actual field w. dynamics

$$Z(J, \bar{\eta}, \eta, \overset{\text{why not??}}{J^\mu_{ext}}) = \int \mathcal{D}\Phi^\dagger \mathcal{D}\Phi \, \mathcal{D}\bar{\psi}\psi \, \mathcal{D}A^\mu \, e^{+iS}$$

$$S = \int d^4x \Big[D_\mu\Phi^\dagger D^\mu\Phi - V(\Phi^\dagger\Phi) + \bar{\psi}(i\not{D}-m)\psi + (\underline{A\text{-only}}) + \Phi J^* + \Phi^* J + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_{ext} A_\mu \Big]$$

If A_μ-only part $=0$, A^μ is just Lagrange multiplier forcing $\bar{\psi}\gamma^\mu\psi = 0$. Really dumb.

But if A_μ-part is interesting - new dynamical theory.

Require: A-only piece unchanged when $A_\mu \to A_\mu + \partial_\mu\theta$ (gauge invariant)

Simplest choice: $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

So put in $\frac{-1}{4e^2} F_{\mu\nu}F^{\mu\nu}$. Common to rescale $A_{old} \to e A_{new}$

$$S = \int d^4x \Big\{ \bar{\psi}(i\not{\partial} + e\not{A} - m)\psi + (\partial_\mu + ieA_\mu)\Phi^*(\partial^\mu - ieA^\mu)\Phi - V(\Phi^*\Phi) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \Big\}$$

This is, as you already saw, the only way to introduce A_μ's into a theory.

Generalization (time permitting)

Suppose multi-component ψ_a, symm $\psi_a \to \psi_a + i\Theta_A T^A_{ab}\psi_b$

i.e. $\psi_a = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$ or $\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix}$ $\quad T^A$ are all traceless Hermitian matrices:

$$T^1 = \frac{1}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix} \quad T^2 = \frac{1}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix} \quad T^3 = \frac{1}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

or $T^{1,2,3} = \frac{1}{2}\begin{bmatrix}\sigma^{123} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0&0\end{matrix} & 0\end{bmatrix}$, $T^4 = \frac{1}{2}\lambda^4 = \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}$ $\lambda^5 = \begin{bmatrix}0&0&-i\\0&&0\\i&0&0\end{bmatrix}$ $\lambda^6 = \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$

$$\lambda^7 = \begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix} \quad \lambda^8 = \begin{bmatrix}1&&\\&1&\\&&-2\end{bmatrix}/\sqrt{3}$$

$$\mathcal{L} = \bar\psi_a(\delta_{ab} i\not\partial + A^A T^A_{ab} - m\delta_{ab})\psi_b \qquad \bar\psi_a \to \bar\psi_a - i\Theta_A T^A_{ca}\bar\psi_c$$

$$\psi_b \to \psi_b + i\Theta_B T^B_{bd}\psi_d$$

$$\mathcal{L} \to (\bar\psi_a - i\Theta_A T^A_{ca}\bar\psi_c)((i\not\partial - m)\delta_{ab} + A^B T^B_{ab} + \delta A^B T^B_{ab})(\psi_b + i\Theta_C T^C_{bd}\psi_d)$$

at $\mathcal{O}(\Theta)$, we have $\delta\mathcal{L} = i\Theta_A\Big\{ \cancel{-\bar\psi_c T^A_{ca}(i\not\partial - m)\delta_{ab}\psi_b}$

$$-i\bar\psi_c T^A_{ca} A^B T^B_{ab}\psi_b + i\bar\psi_a A^B T^B_{ab} T^A_{bd}\psi_d$$

$$\cancel{+\bar\psi_a(i\not\partial - m)T^a_{ad}\psi_d}$$

$$+\bar\psi_a \delta A^B T^B_{ab}\psi_b \Big\} + i\partial_\mu\Theta_A \bar\psi_a i\gamma^\mu T^A_{ab}\psi_b$$

Variation of $\bar\psi, \psi$ cancel for $i\not\partial - m$ terms, except for $\partial_\mu\Theta_A$ as before. But T^A's get tangled with T^B from $A^B T^B$.

Now we need $\quad \delta A^B_\mu T^B = \partial_\mu\Theta_B T^B + iA^B_\mu[T^A, T^B]\Theta_A$

Define $[T^A, T^B] \equiv i f_{ABC} T^C$ structure constants. Totally antisymmetric

works iff $\delta A^A_\mu = \partial_\mu\Theta^A + f_{ABC}A^B_\mu\Theta^C$ new term required.

Now (check!) $F^A_{\mu\nu}F^{\mu\nu}_A$ invariant if $F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + f_{ABC}A^B_\mu A^C_\nu$

$$= i[D_\mu, D_\nu]$$