First serious calculation

$$\mathcal{L}_{QED}[A^\mu, \bar{\Psi}, \Psi] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_e\left(i\gamma^\mu(\partial_\mu + i\overset{q=-1}{q}eA_\mu) - m_e\right)\Psi_e$$

and muons

$$+ \bar{\Psi}_\mu\left(i\gamma^\nu(\partial_\nu + iqeA_\nu) - m_\mu\right)\Psi_\mu$$

sorry, Ψ_{μ} label for name not Lorentz.

Process: $e^+e^- \to \mu^+\mu^-$ one diagram

Momenta: $p, p' \to k, k'$

$q \equiv p + p'$

$$\mathcal{M} = \bar{v}(p', \sigma_{\bar{e}})(ie\gamma^\mu)u(p,\sigma_e)\ \frac{-i\left(g_{\mu\nu} + (\alpha-1)\frac{q_\mu q_\nu}{q^2}\right)}{q^2 + i\varepsilon}\ \bar{u}(k,\sigma_\mu)(ie\gamma^\nu)v(k', \sigma_{\bar{\mu}})$$

$\sigma_e, \sigma_{\bar{e}}, \sigma_\mu, \sigma_{\bar{\mu}}$ spin states

no mass. Note, $q^2 > 0$, $i\varepsilon$ not needed.

Note: $(\bar{u}\gamma^\mu v)^+ = \bar{v}\gamma^\mu u$. why? γ^0, dagger, ... so,

$$\mathcal{M}^*\mathcal{M} = \frac{e^4\left(g_{\mu\nu} + (\alpha-1)\frac{q_\mu q_\nu}{q^2}\right)\left(g_{\alpha\beta} + (\alpha-1)\frac{q_\alpha q_\beta}{q^2}\right)}{(q^2 + i\varepsilon)(q^2 + i\varepsilon)}$$

$$\times\left(\underbrace{\bar{u}_{p\sigma}\gamma^\alpha v_{p'\bar{\sigma}}}_{\#}\ \underbrace{\bar{v}_{p'\bar{\sigma}}\gamma^\mu u_{p\sigma}}_{\#}\right) \times \left(\underbrace{\bar{v}_{k'\bar{\sigma}}\gamma^\beta u_{k\sigma}}_{\#}\ \underbrace{\bar{u}_{k\sigma}\gamma^\nu v_{k'\bar{\sigma}}}_{\#}\right)$$

four #s. I write them in this order "for fun"

~~Can we~~ Do I need $q_\mu q_\nu$? NO. Note: $q_\mu\ \bar{v}_{p'}\gamma^\mu u_p$

$$= \bar{v}(p',\sigma')(\not{q} = \not{p} + \not{p}')u(p,\sigma) = \bar{v}(p',\sigma')(\underbrace{\not{p}' - m}_{0} + \underbrace{\not{p} + m}_{0 \text{ Dirac Eq.}})u(p,\sigma)$$

$$= 0$$

Key: $q = p + p'$ no accident. $J^\mu q_\mu = 0 \ldots$

Spin state - specific answer - hard.

Measuring / generating spin state - also hard.

Don't measure in-spin: Average (50% chance ↑ 50% ↓)

" " out-spin: $\underline{\underline{\text{Sum}}}$ (you see both.)

$$\overline{|\mathcal{M}|^2} \equiv \cancel{\sum_{\sigma_f}\left(\frac{1}{2}\sum_{\sigma_2}\right)} \sum_{\sigma_i,\sigma_f} \left(\frac{1}{2}\right)^{\#\sigma_{in}} \mathcal{M}^*\mathcal{M}_{\sigma_{1i}\dots\sigma_{nf}}$$

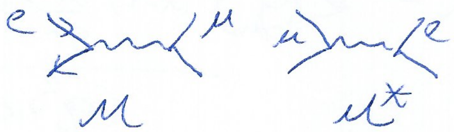
why useful: $\sum_\sigma u(p,\sigma)\bar u(p,\sigma) = \not p + m$

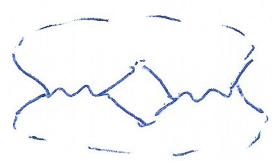
$$\sum_\sigma v(p,\sigma)\bar v(p,\sigma) = \not p - m$$

We had $\bar u \gamma^\alpha v \bar v \gamma^\mu u = \bar u_a \underbrace{\gamma^\alpha_{ab}}_{\text{mat}} \underbrace{v_b \bar v_c}_{\text{matrix}} \underbrace{\gamma^\mu_{cd}}_{\text{mat}} u_d = \mathrm{Tr}\, u_d \bar u_a \gamma^\alpha_{ab} v_b \bar v_c \gamma^\mu_{cd}$ ✓

$$= \mathrm{Tr}\left[(\not p + m)\gamma^\alpha(\not p' - m)\gamma^\mu\right] \times \mathrm{Tr}\left[(\not k' - m_\mu)\gamma^\beta(\not k + m_\mu)\gamma^\nu\right]\left(e^4 \frac{g_{\alpha\beta}\, g_{\mu\nu}}{(q^2)^2}\right)$$

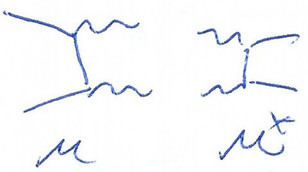
Note: Tr always over elements of $\underline{\text{one}}$ fermion line, followed back-to-front-and-back.

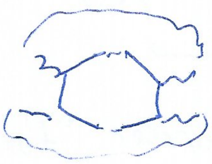


$\mathcal{M}^*\mathcal{M} =$  two ferm. loops. Two traces.

$\underline{\text{vs}}$

$e^+e^- \to \gamma\gamma$



forms  one big F-loop, one big trace.

Computing traces

Cycling $\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \mathrm{Tr}(2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu)$

$$= 2g^{\mu\nu}\underbrace{\mathrm{Tr}\,\mathbb{1}}_{4 \text{ not } 1} - \underbrace{\mathrm{Tr}\,\gamma^\mu\gamma^\nu}_{\mathrm{Tr}\,\gamma^\nu\gamma^\mu = \mathrm{Tr}\,\gamma^\mu\gamma^\nu}$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = 2g^{\mu\nu}\,\mathrm{Tr}\,\gamma^\alpha\gamma^\beta - \mathrm{Tr}\,\gamma^\nu\underbrace{\gamma^\mu\gamma^\alpha}_{\text{do it again}}\gamma^\beta$$

$$= 2g^{\mu\nu}g^{\alpha\beta}\cdot 4 - 2g^{\mu\alpha}\,\mathrm{Tr}\,\gamma^\nu\gamma^\beta + \mathrm{Tr}\,\gamma^\nu\gamma^\alpha\underbrace{\gamma^\mu\gamma^\beta}_{\text{and again}}$$

same as

$$= 2\cdot 4(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}) - \mathrm{Tr}\,\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu$$

odd γ's $\longrightarrow 0$

even γ's $\longrightarrow 4 \times \sum_{\text{pairings}} (-)^{\text{perm}\ldots}\, g\, g\, g \ldots$

~~Some Work~~

$$|\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{s^2}\cdot 4(p^\alpha p'^\mu + p^\mu p'^\alpha - (p\cdot p' + m_e^2)g^{\mu\alpha})\,4(k_\alpha k'_\mu + k_\mu k'_\alpha - (k\cdot k' + m_\mu^2)\times g_{\mu\alpha})$$

Note, $(p+p')^2 = m_e^2 + 2p\cdot p' + m_e^2 = s$. $p\cdot p' = \frac{s}{2} - m_e^2$

Now $s \geq 4m_\mu^2$ (need energy to make 2μ's!) $\frac{m_e^2}{m_\mu^2} = \frac{511\,\mathrm{keV}}{105000\,\mathrm{keV}} \approx \frac{1}{40{,}000}$

(Don't you, forget about m_e ??) Forget m_e.

Some work..

$$|\mathcal{M}|^2 = \frac{8e^4}{s^2}(p\cdot k\, p'\cdot k' + p'\cdot k\, p\cdot k' + 2m_\mu^2\, p\cdot p')$$

$$\sigma = \frac{1}{\underbrace{2E_p 2E_{p'}|v_e - v_{e'}|}_{\simeq 2s}}\int \frac{d^3k\, d^3k'}{(2\pi)^6 2E_k 2E_{k'}}(2\pi)^4\delta^4(p+p'-k-k') \times$$

Near an answer!

CM frame: p-k angle θ

$p^\mu = (E\ 0\ 0\ E)$ $\quad s = 4E^2$

$p'^\mu = (E\ 0\ 0\ -E)$

$k^\mu = (E\ k\sin\theta\ 0\ k\cos\theta)$ $\quad k = \sqrt{E^2 - m_\mu^2}$

$k'^\mu = (E\ -k\sin\theta\ 0\ -k\cos\theta)$

$$\int \frac{d^3k'}{(2\pi)^3} (2\pi)^3 \delta^3(\vec{p} + \vec{p}' - \vec{k} - \vec{k}') = 1$$

$$\int \frac{k^2 dk}{2\pi} 2\pi \delta(2E - 2k^0) = \int k\, k^0 dk^0\, \delta(2E - 2k^0) = \frac{k}{2}$$

Algebra

$$\frac{d\sigma}{d\cos\theta} \propto \left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta \quad \text{and} \quad \sigma_{tot} = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left(1 + \frac{1}{2}\frac{m_\mu^2}{E^2}\right)$$

$\left(E = E_e = E_{\bar{e}} = \frac{1}{2}\sqrt{s} \right)$

0 as $E \to m_\mu$ ✓

$\frac{4\pi\alpha^2}{3E^2}$ as $E \gg m_\mu$ $\quad \alpha = \frac{e^2}{4\pi}$ as usual.

Max at $E = \frac{\sqrt{5}}{\sqrt{\sqrt{21} - 1}} m_\mu$. $\sigma = .0105\,\text{GeV}^{-2} = 4.07 \times 10^{-6}$ barn

(What's a barn? 100 fm^2)

σ, angular pattern at $\leq 1\%$ agreement w. XPT.

Other chg part? Just put in $q^2 = 1 \Rightarrow Q^2$.

Quarks?

$$\frac{\sigma_{\text{all hadrons}}}{\sigma_{\mu^+\mu^-}} \simeq \frac{3\sum Q^2}{1}$$

why 3? Colors
r, g, b pair

$Q^2 = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2$

$+ \left(\frac{2}{3}\right)^2$ $E > 1.5$ GeV

$+ \left(\frac{1}{3}\right)^2$ $E > 5$ GeV

...

Also correct...

We can do more!

Polarize e^+, e^- beams (you can!)

$$\frac{1+\gamma^5}{2} = P_R . \quad \frac{1-\gamma^5}{2} = P_L . \qquad \bar{v}(p')\gamma^\mu u(p)$$

$$= \bar{v}(p')\gamma^\mu (P_L + P_R) u(p)$$

$$= \bar{v}(p') P_R \gamma^\mu P_L u(p) + \bar{v}(p') P_L \gamma^\mu P_R u(p)$$

Only L-e hits R-$\bar{e}$. Le, L$\bar{e}$ beams will have $\sigma_{e\bar{e}\to\mu\bar{\mu}} = 0$

R-e hits L-$\bar{e}$ (really, m^2 suppr.)

why? At rest $u_+ = \sqrt{2m}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $u_- = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\sqrt{2m}$

high-E: $u_+ = \sqrt{2m}\begin{bmatrix}\sqrt{\gamma}\\0\\1/\sqrt{\gamma}\\0\end{bmatrix}$ $u_- = \sqrt{2m}\begin{bmatrix}0\\1/\sqrt{\gamma}\\0\\\sqrt{\gamma}\end{bmatrix}$ down by $1/\gamma$

down by $1/\gamma$

$\gamma \simeq E/m$. Forget comp.

Helicity = Chirality

Polariz ‖

Proj $P_L = \begin{bmatrix}0 & \\ & 0 & \\ & & 1 & \\ & & & 1\end{bmatrix}$ $P_R = \begin{bmatrix}1 & \\ & 1 & \\ & & 0 & \\ & & & 0\end{bmatrix}$ pick or

$u(p, \sigma = +)$ same as $\sum_\sigma P_R u$.

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why? e^- $\longrightarrow$ $\longleftarrow$ e^+ total ang mom $\Longrightarrow$ spin-1

$\Rightarrow$ R $\quad \Rightarrow$ L

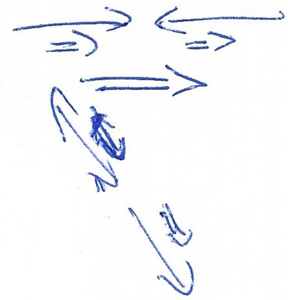
Just like photon

Only $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$. can't get photon w.o. spin-1.

Need $\mathrm{Tr}\, \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \gamma^5 = -4i\, \varepsilon^{\mu\nu\alpha\beta}$

$\mathrm{Tr}\, \gamma^5 = \mathrm{Tr}\, \gamma^\mu \gamma^\nu \gamma^5 = 0$

A little more work

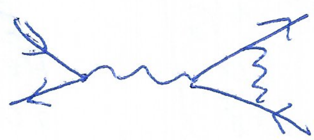


$e^-_R e^+_L \to \mu^-_R \mu^+_L$: $\frac{d\sigma}{d\cos\theta} = (1+\cos\theta)^2$

$\mu^-_L \mu^+_R$ $\quad \frac{d\sigma}{d\cos\theta} = (1-\cos\theta)^2$

QM: Amplitude^2 · spin-1 in z to be spin-1 in $\cos\theta\,\hat{z} + \sin\theta\,\hat{x}$
$\underline{\underline{\text{is}}}$ $(1+\cos\theta)^2$. Just spin QM.

Aside: If $E = m_\mu + a\ \underline{\underline{\text{bit}}}$



$\underline{\text{not}}$ small. Suppressed by α

Enhanced by $\frac{1}{v} = \sqrt{\frac{m}{E-m}}$

for $E-m \sim \alpha^2 m$, $\mathcal{O}(1)$ correcti

That's where bound states, Schrödinger happe