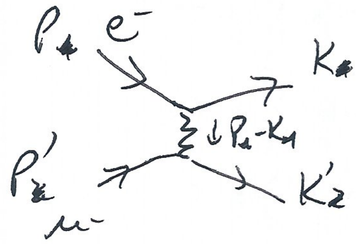
Crossing

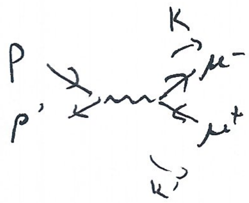
Consider $e^-\mu^- \to e^-\mu^-$



$$\mathcal{M} = \frac{ie^2}{q^2}\,\bar{u}(k,\sigma_f)\gamma^\mu u(p,\sigma_i)\;\bar{u}(k',\sigma_f')\gamma^\nu u(p',\sigma_i')$$

$$|\mathcal{M}|^2 = \frac{e^2 g_{\mu\nu} g_{\alpha\beta}}{4(q^2)^2}\,\mathrm{Tr}\left((\cancel{k}+m_e)\gamma^\mu(\cancel{p}+m_e)\gamma^\alpha\right)\mathrm{Tr}\left((\cancel{k}'+m_\mu)\gamma^\nu(\cancel{p}'+m_\mu)\gamma^\beta\right)$$

|| Exactly || $e^+e^- \to \mu^+\mu^-$ after replacing



$e^+e^- \to \mu^+\mu^-$ $\quad p \to p$ $\quad e^-\mu^- \to e^-\mu^-$

$p' \to -k$

$-k' \to p'$

$-k \to -k'$

Coincidence? No! $u\bar{u}(p) = \cancel{p}+m$

$v\bar{v}(k) = \cancel{k}-m = -(-\cancel{k}+m)$ $\quad p \to k$ works

Therefore $|\bar{\mathcal{M}}|^2 = \dfrac{8e^4}{(q^2)^2}\left(p\cdot k'\, p'\cdot k + p\cdot p'\, k\cdot k' - M_\mu^2\, p\cdot k\right)$

no more computing needed.

But note: $q^2 = -2k^2(1-\cos\theta)$ $\quad$ CM frame $\quad p = (k, k, 0, 0)$

$k = (E, k\cos\theta, k\sin\theta, 0)$ $\quad E = \sqrt{k^2+m_\mu^2}$

~~in $m^2 \to 0$ limit,~~

$$|\bar{\mathcal{M}}|^2 = \frac{2e^4}{k^2(1-\cos\theta)^2}\left((E+k)^2 + (E+k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right)$$

$\propto 1/\theta^4$ at $\theta \to 0$. But you knew that (Rutherford scattering...)

Crossing: Full rule

Permute $+p$ incoming, $-k$ outgoing

eg, $p \to p$ $\qquad (p+p')^2 = s \to (p-k)^2 = t$

$p' \to -k$ $\qquad (p-k)^2 = t \to (p-k')^2 = u$

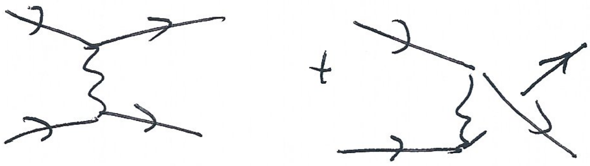
$-k' \to p'$ $\qquad (p-k')^2 = u \to (p+p') = s$

$-k \to -k'$

And (-1) for each fermion switching between in, out state

Interference

Consider $e^- e^- \to e^- e^-$



2 diagrams

$$\mathcal{M} = i(ie)^2 \left[\bar{u}_k \gamma^\mu u_p \bar{u}_{k'} \gamma^\nu u_{p'} \frac{-i g_{\mu\nu}}{(p-k)^2} - \bar{u}_k \gamma^\mu u_{p'} \bar{u}_{k'} \gamma^\nu u_p \frac{-i g_{\mu\nu}}{(p-k')^2} \right]$$

note - sign - already discussed.

$$|\mathcal{M}|^2 = (A^*A) + (B^*B) + \underbrace{(A^*B + B^*A)}_{\text{interference}}$$

$$A^*A = \frac{e^4}{4t^2} \mathrm{Tr}[(\not{k}+m)\gamma^\mu(\not{p}+m)\gamma^\alpha] \, \mathrm{Tr}[(\not{k}'+m)\gamma_\mu(\not{p}'+m)\gamma_\alpha]$$

$$= 2e^4 \left[\frac{s^2+u^2}{t^2}\right] \text{ in } m \to 0 \text{ approx}$$

B^*B : crossing! $k' \leftrightarrow k$. $\quad \frac{s^2+t^2}{u^2} \cdot 2e^4$. $\quad$ But ...

$t \leftrightarrow u$.

Interference:

$$A^*B = -\frac{e^4}{4tu}\,\mathrm{Tr}\left((\not k + m)\gamma^\mu (\not p + m)\gamma^\alpha (\not k' + m)\gamma_\mu (\not p' + m)\gamma_\alpha \right)$$

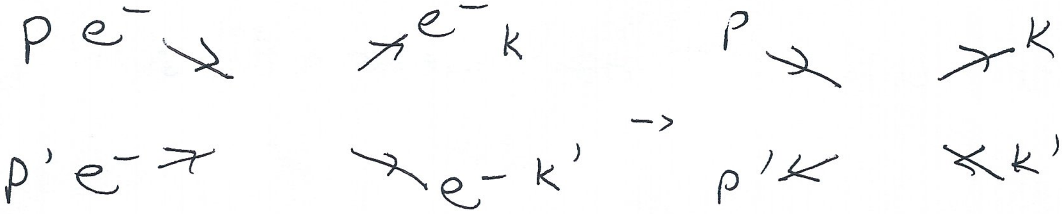
$m \to 0$: need $\mathrm{Tr}\ \underbrace{\not k\, \gamma^\mu \not p\, \gamma^\alpha \not k' \gamma_\mu}_{\text{wait...somehomework...}\ -2\not k' \gamma^\alpha \not p} \not p' \gamma_\alpha$

$= -2\,\mathrm{Tr}\ \not k \not k' \underbrace{\gamma^\alpha \not p \not p' \gamma_\alpha}_{\text{wait...}\ 4p\cdot p'}$ oh, kool. $-32\, p\cdot p'\, k\cdot k'$
$= -8s^2$

$$|\mathcal{M}|^2 = 2e^4\left[\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + 2\frac{s^2}{tu}\right]$$

<u>And</u> only $\int$ over half of final state-space.

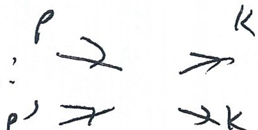
Crossing: $e^+e^- \to e^+e^-$ .... is what?



$p \to p$
$p' \to k'$
$-k \to k$
$-k' \to p'$

$s \to u$
$t \to t$
$u \to s$

Note:



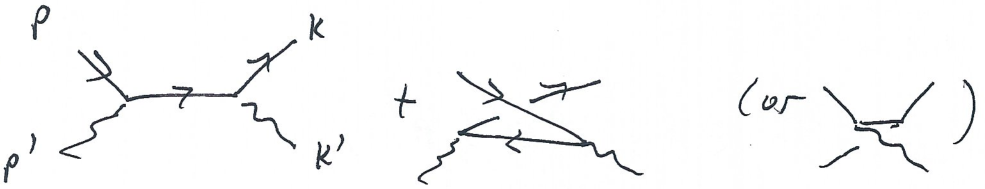
is its <u>own</u> cross
$k \leftrightarrow k'$
$u \leftrightarrow t$

so $|\mathcal{M}|^2$ for $e^-e^- \to e^-e^-$ must be t-u symm. ✓

Photons in final states?

$e^- \gamma \to e^- \gamma$ Compton scattering

2 diagrams



external γ: polarization state $\varepsilon_\mu(\lambda)$ $\quad \varepsilon_\mu(\lambda)\varepsilon^\mu(\lambda') = -\delta_{\lambda\lambda'}$ (why $-$?)

ε^μ in, $\varepsilon^{*\mu}$ out.

$\varepsilon_\mu k^\mu = 0$ $\perp$ to 4-mom.

$$\mathcal{M} = i(-ie)^2 \Big[\bar{u}(k)\, \not{\varepsilon}^*_{\lambda'} \frac{i(\not{p}+\not{p}'+m)}{(p+p')^2-m^2+i\varepsilon} \not{\varepsilon}_\lambda u(p)$$
$$+ \bar{u}(k)\, \not{\varepsilon}_\lambda \frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2+i\varepsilon} \not{\varepsilon}^*_{\lambda'} u(p) \Big]$$

<u>Write</u> as $\varepsilon_\mu \mathcal{M}^\mu$ (factor out ε_μ). [Replace $\not{\varepsilon} \to \gamma^\mu$]

Claim: $p'_\mu \mathcal{M}^\mu = 0$. Why? $A^\mu J_\mu$ again. But seeing it is <u>subtle</u>!

$$\bar{u}(k)\, \not{\varepsilon}^*_{\lambda'} \frac{i(\not{p}+\not{p}'+m)}{(p+p')^2-m^2} \not{p}' u(p) + \bar{u}(k)\, \not{p}' \frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2} \not{\varepsilon}^*_{\lambda'} u(p)$$

add $\not{p} - m$. Dirac Eq.

$(\not{p}+\not{p}'+m)(\not{p}+\not{p}'-m) = (p+p')^2 - m^2$

kills denom.

add $-\not{k}+m$. $\quad p'-k = -p+k'$.

$-(\not{p}-\not{k}'-m)(\not{p}-\not{k}'+m) = -(p-k')^2+m^2$

cancels denom. BUT WITH (-1).

$$i\,\bar{u}(k)\not{\varepsilon}^*_{\lambda'} u(p) - i\,\bar{u}(k)\not{\varepsilon}^*_{\lambda'} u(p) = 0 \quad \text{Aha!}$$

Same for ε^* and k'.

Consider frame, where $p'^{\mu} = [1\,0\,0\,1]E$

Then $\varepsilon_1^{\mu} = [0\,1\,0\,0]$ x-polarization $\quad \varepsilon_\lambda^{\mu}\varepsilon_{\lambda'\mu} = -\delta_{\lambda\lambda'}$ ✓

$\varepsilon_2^{\mu} = [0\,0\,1\,0]$ y-polar. $\quad \varepsilon_\lambda^{\mu} k_\mu = 0$ ✓

$$\sum_\lambda \varepsilon_\lambda^{\mu}\varepsilon_\lambda^{*\nu} = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \neq -g^{\mu\nu}.$$ Schade.

But introduce $\bar{p}'^{\mu} = \frac{1}{2E}[1\,0\,0\,-1]$. $p^{\mu\prime}, \bar{p}'^{\mu}$ form independent null pair of 4-vectors

$\{p'^{\mu}, \bar{p}'^{\mu}, \varepsilon_1^{\mu}, \varepsilon_2^{\mu}\}$ complete set.

Claim: $$g^{\mu\nu} = -\varepsilon_1^{\mu}\varepsilon_1^{\nu} - \varepsilon_2^{\mu}\varepsilon_2^{\nu} - \underbrace{p'^{\mu}\bar{p}'^{\nu} - \bar{p}'^{\mu}p'^{\nu}}$$

$$\begin{bmatrix} 0 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 1/2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1/2 \end{bmatrix}$$

~~But~~

I want $\mathcal{M}^*\mathcal{M} = \varepsilon_\mu^*\mathcal{M}^{\mu *}\varepsilon_\nu\mathcal{M}^\nu = (\varepsilon_\mu^*\varepsilon_\nu)\mathcal{M}^{*\mu}\mathcal{M}^\nu$

If I polariz average, I want $\frac{1}{2}\sum_\lambda \varepsilon_{\mu\lambda}^*\varepsilon_{\nu\lambda}\mathcal{M}^{*\mu}\mathcal{M}^\nu$

Since $p'_\mu\mathcal{M}^{*\mu} = 0 = p'_\nu\mathcal{M}^\nu$, I can put in these!

$\frac{1}{2} g_{\mu\nu}\mathcal{M}^{*\mu}\mathcal{M}^\nu$.

when spin-summing, $\varepsilon^\mu\varepsilon^{\nu *} \rightarrow g^{\mu\nu}$. cool.

Spin & polariz. sum-avg. $\mathcal{M}^\dagger \mathcal{M}$:

$$\frac{e^4}{4}\left(\bar{u}_k\left[\gamma^\nu \frac{i(\not{p}+\not{k}+m)}{s-m^2}\gamma^\mu + \gamma^\mu \frac{i(\not{k}'-\not{p}+m)}{u-m^2}\gamma^\nu\right]u_p\right)$$

$\frac{1}{2}$ e^- spin

$\frac{1}{2}$ γ-polar.

$$\times\left(\bar{u}_p\left[\gamma_\mu \frac{(-i)(\not{p}+\not{p}'+m)}{s-m^2}\gamma_\nu + \gamma_\nu \frac{(-i)(\not{k}'-\not{p}+m)}{u-m^2}\gamma_\mu\right]u_k\right)$$

One big trace.

first term squared, $m \to 0$ limit as example

$$\mathrm{Tr}\left[\underbrace{\not{k}\,\gamma^\nu}(\not{p}+\not{k}')\underbrace{\gamma^\mu \not{p}\, \gamma_\mu}_{-2\not{p}}(\not{p}+\not{p}')\underbrace{\gamma_\nu}\right]$$

8 γ-matrices. 105 terms?

No, use $-2\not{k}$

$$4\,\mathrm{Tr}\left(\not{k}(\not{p}+\not{p}')\not{p}(\not{p}+\not{p}')\right) \qquad \text{But } \not{p}\not{p} = p^2 = m^2 = 0$$

$$4\,\mathrm{Tr}\,\not{k}\not{p}'\not{p}\not{p}' = 16(k\cdot p'\,p\cdot p' + k\cdot p'\,p\cdot p') = -8su \quad \to -2e^4\frac{u}{s} \text{ in } |\bar{\mathcal{M}}|^2.$$

Some work... $|\mathcal{M}|^2 = -2e^4\left(\frac{u}{s}+\frac{s}{u}\right)$. Not bad.

Including m,

$$|\bar{\mathcal{M}}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot p'} + \frac{p\cdot p'}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot p'} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot p'} - \frac{1}{p\cdot k'}\right)^2\right]$$

So now we can do photons...