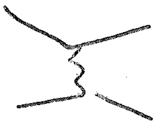
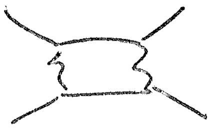
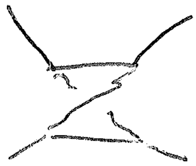
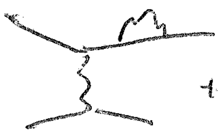
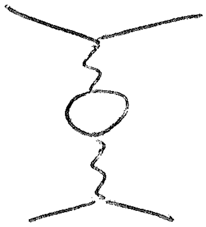
High order processes

Consider $e^- - \mu^-$ scattering. Lowest order  $\propto e^2$

Next order:

 box  crossed box

 +  vertex correction.  +(3) fermion self-energy

 photon self-energy. All $\mathcal{O}(e^4)$

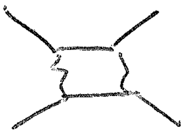
$\mathcal{M} = \mathcal{M}_{e^2} + \mathcal{M}_{e^4}$

$\mathcal{M}^* \mathcal{M} = \underbrace{\mathcal{M}^*_{e^2} \mathcal{M}_{e^2}}_{e^4} + \underbrace{(\mathcal{M}^*_{e^2} \mathcal{M}_{e^4} + \text{h.c.})}_{e^6} + \ldots$

But at e^6 I also need

 +(3) photon emission modified

Let's look at these 1 by 1 (or so)

 computable, finite. Only interesting for $v \ll c$ where it is $\frac{\alpha}{v}$ relative to  ....

L25 p2

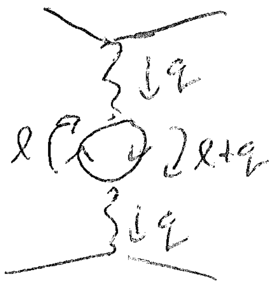


Diagram without: $G_{\mu\nu}(q) = \dfrac{-i(g_{\mu\nu} - q_\mu q_\nu/q^2)}{q^2+i\varepsilon}$

Diagram with:

$$G_{\mu\alpha}(q)\left(i\Pi^{\alpha\beta}(q)\right)G_{\beta\nu}(q)$$

$$i\Pi^{\alpha\beta}(q) = -(ie)^2\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i(\not\ell+m)}{\ell^2-m^2+i\varepsilon}\gamma^\alpha\, i\,\frac{(\not\ell+\not q+m)}{(\ell+q)^2-m^2+i\varepsilon}\gamma^\beta\right]$$

Claim 1: $q_\alpha\Pi^{\alpha\beta}(q)=0$. Proof:

$$q_\alpha\Pi^{\alpha\beta} = +ie^2\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{\not\ell+m}{\ell^2-m^2}(\not q)\frac{(\not\ell+\not q+m)}{(\ell+q)^2-m^2}\gamma^\beta\right]$$

write as $(\not q+\not\ell-m)-(\not\ell-m)$

$(\not q+\not\ell-m)(\not q+\not\ell+m) = (q+\ell)^2-m^2$ cancels denom.

$$q_\alpha\Pi^{\alpha\beta} = ie^2\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\left[-\frac{(\not\ell+\not q+m)}{(\ell+q)^2-m^2}\gamma^\beta + \frac{\not\ell+m}{\ell^2-m^2}\gamma^\beta\right]$$

shift $\int$: $\ell\to\ell-q$ in first term: they cancel!!
$\ell+q\to\ell$

Can I always make such a shift? yes, as long as your theory is gauge invariant. Kinetic momentum $p_\mu-A_\mu$ is physical.

Consider $\theta = \xi^\mu x_\mu$

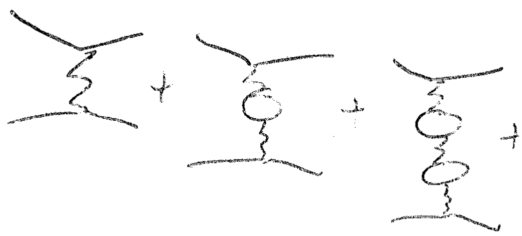
$A_\mu\to A_\mu+\partial_\mu\theta = A_\mu+\xi_\mu$ shifts momentum
$p\to p+\xi$

Regularization (UV issues) must respect such shift if gauge symm. is to survive regularization.

<u>Therefore</u> $\Pi^{\mu\nu}(q) = \Pi(q^2)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$

$\Pi(q^2)$ non<u>singular</u> as $q^2 \to 0$ as m makes integral smooth.

That means $\lim\limits_{q\to 0} \Pi^{\mu\nu}(q) = 0$



$= (\ldots)\Big[\, G_{\mu\nu}(q) + G_{\mu\alpha}\, i\Pi^{\alpha\beta} G_{\beta\nu} + G_{\mu\alpha}\, i\Pi^{\alpha\beta} G_{\beta\gamma}\, i\Pi^{\gamma\delta} G_{\delta\nu} + \ldots$

Now $G_{\mu\nu} = \dfrac{-iP_{\mu\nu}}{q^2}$ $\qquad P_{\mu\nu} \equiv g_{\mu\nu} - q_\mu q_\nu / q^2$

note, projector: $P_{\mu\alpha} P^{\alpha}{}_{\nu} = \left(g_{\mu\alpha} - \dfrac{q_\mu q_\alpha}{q^2}\right)\left(g^{\alpha}_{\nu} - \dfrac{q^\alpha q_\nu}{q^2}\right)$

$= g_{\mu\nu} - \dfrac{q_\mu q_\nu}{q^2}$

$\Pi^{\alpha\beta} = P^{\alpha\beta} q^2 \Pi(q)$

series $= (\,)\, P_{\mu\nu}\Big(\underbrace{\frac{-i}{q^2} + \frac{-i}{q^2}\, iq^2\Pi\, \frac{-i}{q^2} + \frac{-i}{q^2}\, iq^2\Pi\, \frac{-i}{q^2}\, iq^2\Pi\, \frac{-i}{q^2} + \ldots}_{\text{Geometric series! Cool.}}$

$\dfrac{-i}{q^2(1-\Pi(q^2))}$ <u>shift</u> in denominator.

Remark: Always work with $e^2 G_{\mu\nu} = \dfrac{e^2}{q^2(1-\Pi(q^2))} = \dfrac{1}{q^2\left(\frac{1}{e^2} - \frac{\Pi(q^2)}{e^2}\right)}$

$\dfrac{\Pi(q^2)}{e^2}$ is <u>shift</u> in value of $1/e^2$.

Let's get on with it: compute by applying $g_{\mu\nu}\Pi^{\mu\nu} = 3q^2\Pi$

write as $(D-1)q^2\Pi = g_{\mu\nu}\Pi^{\mu\nu}$ $\qquad D = g^{\mu}_{\mu}\ (=4)$

$$(D-1)q^2\Pi = ie^2\int\frac{d^D\ell}{(2\pi)^D}\,\mathrm{Tr}\left[\frac{\not\ell+m}{\ell^2-m^2+i\varepsilon}\gamma^\mu\frac{\not\ell+\not q+m}{(\ell+q)^2-m^2+i\varepsilon}\gamma_\mu\right]$$

uh-oh. at large ℓ, $\dfrac{\mathrm{Tr}\not\ell\gamma^\mu\not\ell\gamma_\mu}{\ell^2\ \ell^2} = \dfrac{-8\ell^2}{\ell^2\ell^2} \sim \dfrac{-8}{\ell^2}$ times $\int d^4\ell$ is way divergent.

Need some UV regularizing procedure. Options

- Cutoff $\int_0^\Lambda d^4\ell$. Nonono. Violates $\int d\ell \to \int d(\ell-q)$ shift. Not gauge invariant.
- Pauli-Villars. Works. Horrible. Avoid learning
- Lattice. Works [after Wick rotating, see below]. Horrible.
- Heat kernel / Schwinger proper time. Equivalent to ...
- Dimensional Regularization.

Imagine I live in $D=4-2\epsilon$ dimensions.
Integrals a little less UV divergent. Makes life better.
Weird but useful method. Learn this!

$$ie^2\underbrace{\mu^{4-D}}\int\frac{d^D\ell}{(2\pi)^D}\,\mathrm{Tr}\,\frac{\not\ell+m\ \gamma^\mu(\not\ell+\not q+m)\gamma_\mu}{(\ell^2-m^2+i\varepsilon)((\ell+q)^2-m^2+i\varepsilon)}$$

so total dimension = 4.
μ^2: explicit scale which must be introduced.

$$\gamma^\mu\gamma_\mu = D$$
$$\gamma^\mu\not\ell\gamma_\mu = (2-D)\not\ell$$
$$\mathrm{Tr}\,\mathbb{1} = 4 \quad \text{(still)}$$

$$q^2\Pi = \frac{ie^2}{D-1}\int\frac{d^D\ell}{(2\pi)^D}\,\underbrace{4\left(\frac{Dm^2+(2-D)\ell\cdot(\ell+q)}{(\ell^2-m^2+i\varepsilon)((q+\ell)^2-m^2+i\varepsilon)}\right)}_{\text{"Grand"}}$$

Dissimilar denom's are a pain.

Combine denominators w. Feynman trick

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{(xA+(1-x)B)^2}$$

in general
$$\frac{1}{A_1 A_2 \cdots A_n} = \int dx_1 \cdots dx_n \, \frac{\delta(x_1+\ldots+x_n-1)\,(n-1)!}{(x_1 A_1 + \ldots + x_n A_n)^n}$$

$$\frac{1}{AB^2} = -\frac{d}{dB}\left(\frac{1}{AB}\right) \text{ doesn't need own expr.}$$

$$\text{"Grand"} = \frac{Dm^2 + (2-D)\,\ell\cdot(\ell+q)}{(\ell^2-m^2+i\varepsilon)((q+\ell)^2-m^2+i\varepsilon)} = \int_0^1 dx \, \frac{Dm^2+(2-D)(\ell^2+\ell\cdot q)}{\left[x\ell^2 - xm^2 + (1-x)(\ell^2+2\ell\cdot q+q^2) + (1-x)m^2 + i\varepsilon\right]^2}$$

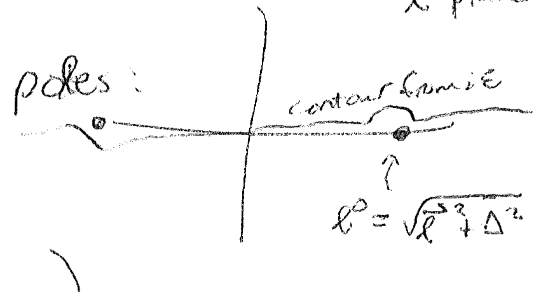
$$= (\ell+(1-x)q)^2 + (x(1-x)q^2 - m^2) + i\varepsilon$$
(note, $q^2<0$.)

shift $\int d^D\ell$: $\ell' = \ell + (1-x)q$

$$q^2\pi = \frac{4}{D-1}\, i e^2 \mu^{4-D} \int_0^1 dx \int \frac{d^D\ell}{(2\pi)^D} \left[\frac{(2-D)(\ell^2 - x(1-x)q^2 + (x-\tfrac{1}{2})q\cdot\ell) + Dm^2}{(\ell^2 - \underbrace{(m^2 - x(1-x)q^2)}_{=\Delta^2} + i\varepsilon)^2}\right]$$

$q\cdot\ell$ term odd in ℓ (and in $x-\tfrac{1}{2}$) – drop it!

Next do
$$\int \frac{d\ell^0}{2\pi} \, \frac{[\ell_0^2 - \ell^2 + \ldots]}{[\ell_0^2 - (\vec{\ell}^2+\Delta^2) + i\varepsilon]^2}$$



Rotate contour
"Wick rotation"
(subtlety: if $\Delta^2<0$)

$$\int \frac{d(i\ell^0)}{2\pi} \, \frac{(i^2\ell_0^2 - \ell^2 + \ldots)}{(i^2\ell_0^2 - \vec{\ell}^2 - \Delta^2)^2}$$

Now define $-\ell_0^2 - \vec{\ell}^2 \equiv \ell_E^2$ (damn metric) ordinary (Euclid.) integration, $g^{\mu\nu} = \begin{bmatrix} 1 & \\ & 1 \\ & & 1 \end{bmatrix}$

$$q^2\pi = \frac{4i^{-2}e^2\mu^{4-D}}{D-1}\int_0^1 dx \int \frac{d^D l_E}{(2\pi)^D} \frac{(2-D)(-l_E^2 - x(1-x)q^2) + Dm^2}{(-l_E^2 - \Delta^2)^2}$$

Write $\displaystyle\int \frac{d^D l_E}{(2\pi)^D} = \int l^{D-1} dl \int \frac{d\Omega_{D-1}}{(2\pi)^D}$

$\hookrightarrow$ area of S^{D-1} is what?

Well, $\displaystyle\int d^D x\, e^{-x^2/2} = \int x^{D-1} e^{-x^2/2} dx \int d\Omega_{D-1} = 2^{\frac{D-2}{2}}\Gamma(D-2)\int d\Omega_{D-1}$

But it's also $\displaystyle\left(\int dx\, e^{-x^2/2}\right)^D = \left(\sqrt{2\pi}\right)^D$.

$$\boxed{\text{So } \int d\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}}$$ Analytic function so it's unique!

And $\displaystyle\int l^{D-1} dl = \frac{1}{2}\int (l^2)^{\frac{D-2}{2}} d(l^2)$.

$\displaystyle\int y^n (y+\Delta)^{-m} dy = \Delta^{n+1-m}\int_0^1 dx\, x^{m-n-2}(1-x)^n$ $(x = \Delta/_{y+\Delta})$. $\displaystyle\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ "Beta func"

$$\int \frac{l^{2x-1}\, dl}{(l^2+\Delta)^y} = \frac{\Delta^{x-y}\Gamma(x)\Gamma(y-x)}{2\Gamma(y)}$$ wol. EG, $2x = D$; $y = 2$: $\displaystyle\boxed{\Delta^{\frac{D}{2}-2}\frac{\Gamma(D/_2)\Gamma(2-D/_2)}{2\Gamma(2)}}$

Now log divergences $\to \Gamma\left(\frac{4-D}{2}\right)$. What is this?

And what's $\Delta^{\frac{D}{2}-2}$? Name $D = 4-2\varepsilon$ (some people use $4-\varepsilon$ - beware)

$$\Gamma\left(\tfrac{4-D}{2}\right) = \Gamma(\varepsilon) = \int_0^\infty x^{-1+\varepsilon} e^{-x} dx = \frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)$$

$\hookrightarrow$ someday you will need these. Not today.

γ_E: Euler-Mascheroni constant

$\displaystyle\frac{1}{\varepsilon} \overset{???}{=} \mu^{2\varepsilon}\int \frac{d^{4-2\varepsilon} l}{(l^2+\mu^2)^2} \sim \log\left(\frac{\Lambda^2}{\mu^2}\right)$ where Λ is scale at which $\mu^{2\varepsilon} l^{3-2\varepsilon}$ really $\approx l^3$

$\left(\mu/\Lambda\right)^{2\varepsilon} \sim e^{-1} \to \Lambda \sim \mu e^{1/2\varepsilon}$

$\log\left(\frac{\Lambda^2}{\mu^2}\right) \sim \log\left(e^{1/\varepsilon}\right) \sim 1/\varepsilon$ ✓

L25
PG 1/2

Let's talk about divergences

$$\frac{1}{(2\pi)^D}\int \frac{d^D\ell\; \ell^2}{(\ell^2+\Delta)^2} \quad \left(= -\Delta^2\int\frac{d^D\ell}{(\ell^2+\Delta^2)^2} + \int \frac{d^D\ell}{(\ell^2+\Delta^2)^1}\right)$$

Contains log & power divergences

$$\frac{1}{(2\pi)^D}\int\frac{d^D\ell}{(\ell^2+\Delta^2)^1} = \frac{1}{(4\pi)^{D/2}}\int \frac{(\ell^2)^{\frac{D}{2}-1}\, d(\ell^2)}{(\ell^2+\Delta^2)^1} = \frac{1}{(4\pi)^{D/2}}\, \Delta^{D-2}\frac{\Gamma(D/2)\,\Gamma(1-D/2)}{\Gamma(1)}$$

Divergent because of $\Gamma(1-D/2) = \Gamma(1-2+\varepsilon) = \Gamma(\varepsilon-1)$

$\Gamma(x)$ diverges at $x=0, -1, -2, \dots$

First divergent at $D=2$, not $D=4$. Indicates quadratic div.

But note: in our problem, it has a prefactor $(2-D)$

$$(2-D)\,\Gamma(1-D/2) = 2(1-D/2)\,\Gamma(1-D/2)$$
$$= 2\,\Gamma(2-D/2) \quad \text{actually first diverges for } D=4.$$

This is how Dim Reg tells us our theory doesn't really have power divergences.

Compare to scalar $\frac{\lambda}{24}\varphi^4$ theory.

 this diagram $\rightarrow \frac{\lambda^2}{2}\int \frac{d^D\ell\,(\mu^{4-D})}{(2\pi)^D}\,\frac{1}{(\ell^2-m^2+i\varepsilon)}\,\frac{1}{((\ell+q)^2-m^2+i\varepsilon)}$

$$\rightarrow \int_0^1 dx\, \frac{\lambda^2\mu^{4-D}}{(4\pi)^{D/2}}\int \frac{\ell_E^{D-1}\,d\ell_E}{(\ell_E^2+\Delta^2)^2}$$

same meaning for Δ^2.
Log div. only.

But 
self energy corr

$$\frac{\lambda}{2}\int\frac{d^D\ell\,\mu^{4-D}}{(2\pi)^D}\,\frac{1}{(\ell^2-m^2+i\varepsilon)} \rightarrow \frac{\lambda\mu^{4-D}}{2(4\pi)^D}\int \frac{\ell_E^{D-1}\,d\ell_E}{(\ell_E^2+m^2)^1}$$

really does have $\Gamma(\varepsilon-1)$, really diverges in 2D, really is quadratically divergent – though div. is q-independent (~~at every loop order~~!)

Also $(\Delta/\mu^2)^\varepsilon = \cancel{e^{\varepsilon \ln(\Delta/\mu^2)}}$

$$(\mu^2/\Delta)^\varepsilon = e^{\varepsilon \ln(\mu^2/\Delta)} = 1 + \varepsilon \ln(\mu^2/\Delta) + \mathcal{O}(\varepsilon^2)$$

I need $\mathcal{O}(\varepsilon)$ next to the $1/\varepsilon$ term.

$$\Pi(q) \overset{\text{Dust settles}}{=} \frac{-e^2}{12\pi^2} \Gamma(\varepsilon) \int_0^1 dx\, 6x(1-x) \left[\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right]^\varepsilon$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon}\Gamma(1+\varepsilon) = \frac{1}{\varepsilon}\left(\Gamma(1) + \varepsilon\Gamma'(1) + \dots\right)$$

$$= \frac{1}{\varepsilon}(1 - \varepsilon\gamma_E) = \frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(1/\varepsilon)$$

Need remaining term to $\mathcal{O}(\varepsilon)$. $\left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)^\varepsilon = 1 + \varepsilon \ln(4\pi\mu^2)$
$- \varepsilon \ln(m^2 - x(1-x)q^2)$

$$\Pi(q^2) = \frac{-e^2}{12\pi^2}\left(\underbrace{\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)}_{= 1/\bar{\varepsilon}} + F\left(\frac{\mu^2}{m^2}, \frac{\mu^2}{q^2}\right)\right)$$

$$F = \int_0^1 dx\, 6x(1-x) \ln \frac{\mu^2}{m^2 - x(1-x)q^2} \to \begin{cases} \ln\frac{\mu^2}{m^2} & q^2 \ll m^2 \\ \ln\frac{\mu^2}{q^2} - 5/3 & q^2 \gg m^2 \end{cases}$$

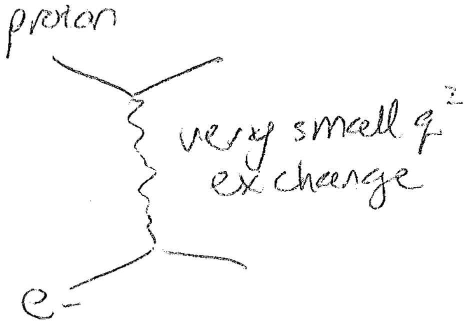
Scattering involves $$\frac{e^2}{1 + \frac{e^2}{12\pi^2}\left(\frac{1}{\bar{\varepsilon}} + F\right)} = \frac{1}{\left(\frac{1}{e^2} + \frac{1}{12\pi^2}\frac{1}{\bar{\varepsilon}}\right) + F/12\pi^2}$$

$1/\bar{\varepsilon}$ still in there. Answer depends on UV cutoff $1/\bar{\varepsilon}$.
Also depends on unknown charge $1/e^2$, the Lagrangian parameter eg, $\frac{-1}{4e^2}F_{\mu\nu}F^{\mu\nu}$

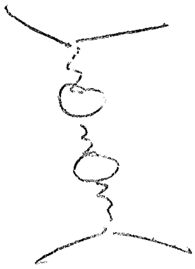
Renormalization

When / how did I measure e^2 anyways?

By looking at atoms / long range EM forces



also contains



What I measure and call e^2_{phys} is really

$$e_{phys}^{-2} = \underset{\substack{\text{bare, } \mathcal{L} \\ \text{version}}}{e_0^{-2}} + \frac{1}{12\pi^2}\frac{1}{\varepsilon}$$

: Only quantity I actually know. e_0^{-2} depends on $1/\varepsilon$ but e_{phys}^{-2} does not.

Scattering: $$e^2 \to \frac{1}{e_{phys}^{-2} + \frac{1}{12\pi^2}\left[F\left(\frac{\mu^2}{m^2}, \frac{\mu^2}{q^2}\right) - F(q \to 0)\right]}$$

This is renormalization: when I compare physical processes, sensitivity to UV cancels between divergencies & relation between $\mathcal{L}$ param's and measured param's