Nöther Theorem

Our example φ_a $a=1..N$, $\mathcal{L}=\frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a-\frac{\omega^2}{2}\varphi_a\varphi_a-\frac{\lambda}{8}\varphi_a\varphi_a\varphi_b\varphi_b$

has invariance: $\forall$ pair $a>b$, we can take $\varphi_a\to\varphi_a+\varepsilon\varphi_b$

$\varphi_b\to\varphi_b-\varepsilon\varphi_a$ and $\mathcal{L}$ unchanged (to lin. order in ε)

Name these $\frac{(N)(N-1)}{2}$ transforms Transform # A $A=1,\dots,\frac{N(N-1)}{2}$

Most general trans. is $\varphi_a\to\varepsilon_A\, i\, T_A^{ab}\varphi_b$

$$T_1^{ab}=\begin{bmatrix}0&-i&\\ i&0&\\ &&\ddots\end{bmatrix}\qquad T_2^{ab}=\begin{bmatrix}0&0&-i\\ 0&&\\ i&&\end{bmatrix}\quad \text{etc.}$$

(i chosen so T^{ab} a Hermitian matrix)

ε_A: "angles" by which I rotate. (How much to rotate)

T_A^{ab} "generators" of trans. of φ_a. How each rotation works

Replacement $\varphi_a\to\bar{\varphi}_a=\varphi_a+i\epsilon_A T_A^{ab}\varphi_b$ takes $\mathcal{L}(\varphi)\to\mathcal{L}(\bar{\varphi})$

$=\mathcal{L}(\varphi)$ unchanged (in value)

More generally it might happen that symm takes

$$\mathcal{L}(\varphi)\to\mathcal{L}(\bar{\varphi})=\mathcal{L}(\varphi)+\varepsilon_A\partial_\mu\mathcal{J}_A^\mu(\varphi)\quad \mathcal{J}_A^\mu \text{ some extra stuff}$$

But in this case $\int d^4x\,\mathcal{L}(\varphi)\to\int d^4x\,\mathcal{L}(\bar{\varphi})$

$$=\int d^4x\,\mathcal{L}(\varphi)+\underbrace{\int d^4x\,\varepsilon_A\partial_\mu\mathcal{J}_A^\mu}_{\text{stupid bdry term}}=0$$

Let's see what we learn by working out explicitly what change to $\mathcal{L}$ looks like.

$$0 = \mathcal{L}(\bar{\varphi}) - \mathcal{L}(\varphi) - \varepsilon_A \partial_\mu J^\mu_A$$

→ in case $\mathcal{L}$ changes by total deriv rather than being unchanged.

$$0 = i\varepsilon_A T_A^{ab} \varphi_b \frac{\delta \mathcal{L}}{\delta \varphi_a} + i\varepsilon_A T_A^{ab} (\partial_\mu \varphi_b) \underbrace{\frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi_a}}_{= \pi^\mu_a} + \cancel{\mathcal{L}(\hat{\varphi})} - \cancel{\mathcal{L}(\varphi)} - \varepsilon_A \partial_\mu J^\mu_A$$

write $(\partial_\mu \varphi_b)\, \pi^\mu_a = \partial_\mu(\varphi_b \pi^\mu_a) - \varphi_b \partial_\mu \pi^\mu_a$

Oh! and $\partial_\mu \pi^\mu_a = \frac{\delta \mathcal{L}}{\delta \varphi_a}$! so these terms cancel by eq. of motion

$$0 = \varepsilon_A \left[i T_A^{ab} \partial_\mu (\pi^\mu_a \varphi_b) - \partial_\mu J^\mu_A \right] = \varepsilon_A \partial_\mu \underbrace{\left[i T_A^{ab} \varphi_b \pi^\mu_a - J^\mu_A \right]}_{\text{name it } J^\mu_A}$$

True iff $\partial_\mu J^\mu_A = 0$ as ε_A are arbitrary angles

Conserved current J^μ_A. [For our example, $\pi^\mu_a \varphi_b - \pi^\mu_b \varphi_a \quad \forall a > b$]

Define $Q_A = \int d^3x\, J^0_A$.

$$\partial_0 Q_A = \int d^3x\, \partial_0 J^0_A = \int d^3x\, -\nabla_i J^i.$$

is total space deriv → 0

Q_A is unchanged with time.

$Q_{ab} = -\varphi_a \dot{\varphi}_b + \varphi_b \dot{\varphi}_a$ are like "field-space angular momenta"

What about *actual* energies & angular momenta?

We know that if I shift $\varphi(x) \to \varphi(x+\xi)$ translation

then $\int d^4x\, \mathcal{L}(\varphi(x)) = \int d^4x\, \mathcal{L}(\varphi(x+\xi))$ by shifting coord $y = x - \xi$ ....

So that's a symmetry. Let's see how this works.

$$\varphi(x+\xi) = \varphi(x) + \xi^\mu \partial_\mu \varphi(x) + \mathcal{O}(\xi^2) \quad \text{Taylor}$$

$$\mathcal{L}(\varphi+\xi) = \mathcal{L}(\varphi) + \xi^\mu \partial_\mu \mathcal{L}$$

not unchanged. So what is my "$\mathcal{J}^\mu$" ??

Confusing. μ here is like "A" keeping track of which transform I am making.

I want to write $\delta\mathcal{L} = \xi^A \partial_\mu \mathcal{J}^\mu_A$

this is the ∂_μ above... $\mathcal{J}^\mu_A = g^\mu_A \mathcal{L}$

$$\mathcal{L}(\varphi+\xi) = \mathcal{L}(\varphi) + \xi^\mu \partial_\nu (g^\nu_\mu \mathcal{L})$$

Conserved current J^ν_μ with $\partial_\nu J^\nu_\mu = 0$ is

$$J^\nu_\mu \equiv \boxed{T^\nu_\mu = \pi^\nu \partial_\mu \varphi - g^\nu_\mu \mathcal{L}} \quad \text{Stress Tensor}$$

Note: ν: direction of flow: time or space dir.
μ: type of charge: energy or momentum.

T^0_0 : time flow of energy $= H$

T^i_0 : space flux of energy

T^0_i : time flow of momentum

T^i_j : space flux of momentum

$$P^\mu \equiv \int d^3x\, T^{0\mu} \quad \text{total 4-momentum}$$

$\partial_\mu T^{\mu\nu} = 0$ energy-momentum conservation

Lorentz transformation

Translation invariance is statement

Physics unchanged by replacement $\varphi(x^\mu) \to \varphi(x^\mu + \xi^\mu)$

a translation $x \to x + \xi$

Lorentz: $X^\mu \to \cancel{x}\,\Lambda^\mu{}_\nu X^\nu$ $\quad \Lambda^\mu{}_\nu$ a Lorentz transform.

What $\Lambda^\mu{}_\nu$ are allowed? Must ~~con~~preserve invariant length

$$X^\mu \to \bar{X}^\mu = \Lambda^\mu{}_\nu X^\nu \text{ has } g_{\mu\nu} X^\mu X^\nu = g_{\mu\nu} \bar{X}^\mu \bar{X}^\nu$$

$$(t^2 - \vec{x}^2 = \bar{t}^2 - \bar{\vec{x}}^2)$$

Let's work that out for infinitesimal

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \underbrace{\omega^\mu{}_\nu}_{\text{some small shifts}} \qquad \bar{X}^\mu = X^\mu + \omega^\mu{}_\nu X^\nu$$

$$g_{\mu\nu} \bar{X}^\mu \bar{X}^\nu = g_{\mu\nu} (X^\mu + \omega^\mu{}_\alpha X^\alpha)(X^\nu + \omega^\nu{}_\beta X^\beta)$$

$$= g_{\mu\nu} X^\mu X^\nu + g_{\mu\nu} \omega^\mu{}_\alpha X^\alpha X^\nu + g_{\mu\nu} X^\mu \omega^\nu{}_\beta X^\beta + \mathcal{O}(\omega^2)$$

$$= g_{\mu\nu} X^\mu X^\nu + \cancel{g}(\omega_{\nu\mu} + \omega_{\mu\nu}) X^\mu X^\nu$$

Requirement $g_{\mu\nu} X^\mu X^\nu = g_{\mu\nu} \bar{X}^\mu \bar{X}^\nu$ is requirement $(\omega_{\nu\mu} + \omega_{\mu\nu}) X^\mu X^\nu = 0 \quad \underline{\forall X}$

$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$ or $\omega_{\nu\mu} = -\omega_{\mu\nu}$

antisymmetry

Careful: $\omega_{\mu\nu}$ antisymm, $\omega^\mu{}_\nu$ is not!

$$\omega^\mu{}_\nu = \begin{bmatrix} 0 & b_x & b_y & b_z \\ b_x & 0 & -r_x & r_y \\ b_y & r_x & 0 & -r_z \\ b_z & -r_y & r_z & 0 \end{bmatrix} \text{ is } \omega_{\mu\nu} = \begin{bmatrix} 0 & b_x & b_y & b_z \\ -b_x & 0 & r_x & -r_y \\ -b_y & -r_x & 0 & r_z \\ -b_z & r_y & -r_z & 0 \end{bmatrix} \text{ antisymm matrix}$$

reminder: if $b_x \neq 0$,

$$\omega^\mu{}_\nu = \begin{bmatrix} 0 & b_x & 0 & 0 \\ b_x & & & \\ 0 & & 0 & \\ 0 & & & \end{bmatrix}, \quad \Lambda^\mu{}_\nu = \exp(\omega^\mu{}_\nu)$$

$$= \begin{bmatrix} \cosh b_x & \sinh b_x & 0 & 0 \\ \sinh b_x & \cosh b_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ boost}$$

if

$$\omega^\mu{}_\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r_x & 0 \\ 0 & r_x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Lambda^\mu{}_\nu = \exp(\omega^\mu{}_\nu)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos r_x & -\sin r_x & 0 \\ 0 & \sin r_x & \cos r_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ rotation}$$

If $\vec{b}, \vec{r}$ both nonzero, $\Lambda^\mu{}_\nu$ is something complicated. But can always be expressed as "rotation times a boost"

Associated Nöther current?

$$x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$$

$$\varphi(x) \to \varphi(x) + \underbrace{(\omega^\mu{}_\nu x^\nu)}_{\text{space-dependent translation}} \partial_\mu \varphi(x)$$

$\partial_\mu \varphi(x)$: response to translation

$$\mathcal{L}(\varphi) \to \mathcal{L}(\varphi) + \omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = \mathcal{L}(\varphi) + \omega^\mu{}_\nu x^\nu \partial_\alpha (g^\alpha_\mu \mathcal{L})$$

$$= \partial_\alpha (x^\nu g^\alpha_\mu \mathcal{L})$$

cos $\partial_\alpha x^\nu = g^\nu_\alpha$, $\omega^\mu{}_\nu g^\nu_\mu = 0$.

$\mu\nu$: which Lorentz transform

α: $\partial_\alpha \mathcal{J}^\alpha$ thing.

Transform $\omega_{\mu\nu}$ has: $\varphi \to \varphi + \omega_{\mu\nu} x^\nu \partial^\mu \varphi$

$$\mathcal{L} \to \mathcal{L} + \partial_\alpha \mathcal{J}^\alpha, \quad \mathcal{J}^\alpha = x^\nu g^{\mu\alpha} \mathcal{L}$$

$\partial_\alpha M^{\mu\nu\alpha} = 0$, with $\boxed{M^{\mu\nu\alpha} = x^\nu \partial^\mu \varphi \pi^\alpha - x^\nu g^{\mu\alpha} \mathcal{L}}$

or technically $(x^\nu \partial^\mu \varphi - x^\mu \partial^\nu \varphi) \pi^\alpha - (x^\nu g^{\mu\alpha} - x^\mu g^{\nu\alpha}) \mathcal{L}$

what are these? $\int M^{\mu\nu 0}\, d^3x = M^{\mu\nu}$ is conserved

$M^{ij} = \int x^j (\partial^i \varphi\, \pi^0)$ antisymmetrized

↳ oh! That's T^{i0} = momentum density, p^i

$M^{ij} = \int p^i x^j\, d^3x = \cancel{\frac{1}{2}\int d^3x\,(x^i p^j - p^i x^j)}$ really $\int d^3x \left(\frac{p^i x^j - p^j x^i}{2}\right)$

Angular momentum!

M^{0i} = some strange generalization of angular momentum

$= \int d^3x \left(x^0 T^{0i} - x^i T^{00} \right)$

time × momentum − center-of-mass of energy

is conserved. ~~Center of Mass of Energy = time · momentum~~

$\frac{d}{dt}$(center-of-mass of energy) = momentum. sounds kinda obvious??