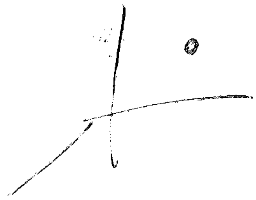
Quantum field theory

Recall quantum mechanics:

Classical thy

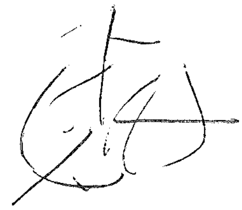
State $\vec{x}$ 



momentum $\vec{p}$ related to $\frac{dx}{dt}$

Quantum thy

Wave function $\psi(\vec{x})$



momentum operator $\hat{p}_i = -i\frac{\partial}{\partial x_i}$

acts on $\psi(x)$ which has momentum info contained in it.

QFT

Same story - except $\vec{x}$ replaced by $\varphi(x)$

Classical

State $\varphi(x)$

momentum $\pi(x)$

function on continuous space

Quantum

$\Psi(\varphi(x))$

$\pi(x) = -i\frac{\partial}{\partial\varphi(x)}\Psi(\varphi(x))$

function over functions of continuous space
("2'nd Quantization")

function in continuously infinitely many dimensions??
Does that make sense? And does $\frac{\partial}{\partial\varphi(x)}$ exist??

Not clear at all!!

Functional analysis we need to make sense of such things does not exist! Not clear $\Psi(\varphi(x))$ as such makes sense!

But: Sufficient to choose some min distance a [say, 10^{-20} m ??]

maximum size L [say, 10^{12} lt yr]

$= 10^{28}$ m

and ~~work~~ work in a box $L \times L \times L$

with discrete spacing a, eg, $10^{48} \times 10^{48} \times 10^{48}$ points

Classically

$\varphi(x)$ is $\varphi(x_1), \varphi(x_2), \varphi(x_3), \ldots$

$\varphi(x_{1437\ldots \times 10^{144}})$

Quantum

$\Psi(\varphi(x))$ is function on 10^{144}-dimensional space.

We are not going to solve Schrödinger eq. on 10^{144}-dim space!

Also - we don't really want these answers - we want limit of answer as $a \to 0$.

Value at finite a: actually well posed (really hard!!) QM problem

Limit as $a \to 0$: how are we taking this limit?

Not so obvious. Many subtleties will later emerge here. But this term: don't worry.

How bad can it be? Let's try it!!

Simplest theory: 1 real scalar field $\varphi(x)$

Classical configuration: $\varphi(x)$ funct. of $\vec{x}$ (or value at each pt x)

Lagrangian $L = \int d^3x \left(\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2\right)$

No $\lambda\varphi^4$ to make life easy!

Let's Solve this!! For now: replace above with

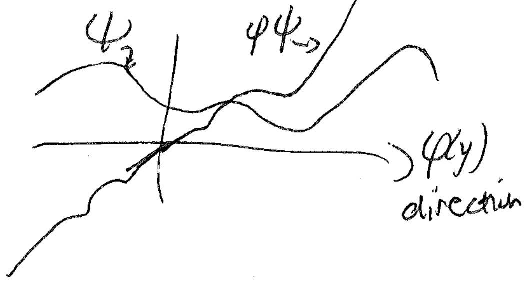
$$L = \sum_{n_1,n_2,n_3=1}^{L/a} a^3\left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2\right]$$

$\hookrightarrow$ $\partial_0\varphi$ makes sense. $\partial_i\varphi$??

something like $\dfrac{\varphi(x+a\hat{i}) - \varphi(x)}{a}$

Note: φ is an operator $\hat{\varphi}$!!

$$\hat{\varphi}(y)\,\Psi(\varphi(x)) = \varphi(y)\,\Psi(\varphi(x))$$



Canonical momentum

$$\pi(\vec{n}) = \frac{\partial L}{\partial\,\partial_0\varphi(\vec{n})} = a^3\,\partial_0\varphi(\vec{n})$$

$\vec{n}$ or $\vec{x}$ whichever you prefer.

$$H = \sum_n \pi\,\partial_0\varphi - L = \sum_n \left(a^{-3}\,\frac{1}{2}\,\pi_{\vec{n}}\pi_{\vec{n}} + \frac{a^3}{2}|\nabla\varphi|^2 + \frac{a^3 m^2}{2}\varphi_{\vec{n}}^2\right)$$

note a's !!

Commutation: if $\pi_{\vec{n}} = -i\dfrac{\partial}{\partial\varphi_{\vec{n}}}$ then $[\pi_{\vec{n}}, \varphi_{\vec{m}}] = -i\,\delta^3_{\vec{n},\vec{m}}$

Kronecker delta.

More usual to introduce $\pi(\vec{x}) = a^{-3}\pi_{\vec{n}}$

"continuum-normalization canonical momentum"

$$H = \sum_{\vec{n}} a^3 \left[\frac{1}{2}\pi^2(x) + \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{m^2}{2}\varphi^2(x)\right]$$

and $[\pi(x), \varphi(y)] = a^{-3}\cdot -i\delta_{x,y}$ " $= -i\,\delta^3(x-y)$ "

Dirac delta

in the sense that $\sum_x a^3 \downarrow = 1$ and $\sum_x a^3 \Rightarrow \int d^3x$

In this notation, $a \to 0$ limit is "almost" harmless.

Now $\nabla_i\varphi(x) = \dfrac{\varphi(x+\hat{i}a) - \varphi(x)}{a}$ is non-diagonal

makes H a non-diagonal matrix operator over the φ's ...

Best use Fourier techniques to fix this!!

Define $$\tilde{\varphi}(p) = \int_0^L d^3x\, e^{-i\vec{p}\cdot\vec{x}}\varphi(x)$$ note, $\vec{p} = \vec{n}\,\frac{2\pi}{L}$

Periodic Bdy...

$$\varphi(x) = \sum_{p = \vec{n}\frac{2\pi}{L}} L^{-3}\tilde{\varphi}(p)e^{+i\vec{p}\cdot\vec{x}}$$ for now I will leave L finite (will be convenient)

$\tilde{\pi}(p)$ defined similarly.

Note: $\tilde{\varphi}^*(p) = \int d^3x\, e^{+ip\cdot x}\varphi(x) = \tilde{\varphi}(-p)$

not $\tilde{\varphi}(p)$.

Now

$$\int d^3x \, \frac{\pi(x)\pi(x)}{2}$$

$$= \sum_{p_1 = n_1 \frac{2\pi}{L}} \sum_{p_2 = n_2 \frac{2\pi}{L}} L^{-6} \int d^3x \, \underbrace{e^{i\vec{p}_1\cdot\vec{x}} \, e^{i\vec{p}_2\cdot\vec{x}}}_{\text{only } x\text{-dependence.}} \, \frac{\tilde{\pi}(p_1)\tilde{\pi}(p_2)}{2}$$

Do the $\int d^3x$ integral!! $L^3 \delta^3_{n_1+n_2}$ forces $n_2 = -n_1$ (note!)

$$= \sum_{p_1 = n\frac{2\pi}{L}} L^{-3} \frac{\tilde{\pi}(p_1)\tilde{\pi}(-p_1)}{2}$$

Similarly, other terms $\rightarrow$ $\displaystyle\sum_{p_1 = n\frac{2\pi}{L}} L^{-3} (p_1^2+m^2) \frac{\tilde{\varphi}(p_1)\tilde{\varphi}(-p_1)}{2}$

$$H = \sum_{p = n\frac{2\pi}{L}} L^{-3} \left[\frac{\tilde{\pi}(p_1)\tilde{\pi}(-p_1)}{2} + (p_1^2+m^2) \frac{\tilde{\varphi}(p_1)\tilde{\varphi}(-p_1)}{2} \right]$$

Looks a lot like many SHO's. It is!! $\omega_p \equiv \sqrt{p^2+m^2}$

Define $a_p = \dfrac{1}{\sqrt{2L^3}} \left(\sqrt{\omega_p}\, \tilde{\varphi}(p) + \dfrac{i}{\sqrt{\omega_p}} \tilde{\pi}(p) \right)$

$a_p^\dagger = \dfrac{1}{\sqrt{2L^3}} \left(\sqrt{\omega_p}\, \tilde{\varphi}(-p) - \dfrac{i}{\sqrt{\omega_p}} \tilde{\pi}(-p) \right)$ note signs

$$H = \sum_{p = \vec{n}\frac{2\pi}{L}} \left(\frac{a_p a_p^\dagger}{2} + \frac{a_{-p}^\dagger a_{-p}}{2} \right) \omega_p = \sum_{p = \vec{n}\frac{2\pi}{L}} (a_p^\dagger a_p + 1/2)\,\omega_p$$

10^{144} Simple Harmonic Oscillators!!

Alternatively we can write

$$\tilde{\varphi}(p) = \sqrt{\frac{L^3}{2\omega_p}}\,(a_p + a_{-p}^\dagger)$$

$$\varphi(x) = \sum_{p = n\frac{2\pi}{L}} \frac{i}{\sqrt{2\omega_p L^3}} \left(e^{i\vec{p}\cdot\vec{x}} a_p + e^{-i\vec{p}\cdot\vec{x}} a_p^\dagger\right)$$

both Hermitian ✓

$$\pi(x) = \sum_{\vec{p} = \frac{2\pi}{L}\vec{n}} \sqrt{\frac{\omega_p}{2L^3}} \left(-ie^{i\vec{p}\cdot\vec{x}} a_p + ie^{-i\vec{p}\cdot\vec{x}} a_p^\dagger\right)$$