So a_p creates a particle of momentum p

But it's spread over whole Universe. Not something you can actually do.

You can make something local in both x and p space

Consider $f(x)$ ~~mmm~~ with dual $\tilde{f}(p)$.

Single-particle state $|f\rangle = \sum_{\vec{p}=\frac{2\pi}{L}\vec{n}} \tilde{f}(p)\, a_p^\dagger |0\rangle$ sensible.

Normalization: $\sum_p \tilde{f}^* \tilde{f}(p) = 1$. Each $\tilde{f}(p) \to 0$ as $L \to \infty$ ~~for some fixed $f(x)$. Sad.~~

Means function $f(x)$ must get smaller as I make L bigger???

I would like to change normalization convention

so things will be written in terms of

$$\sum_{p=\frac{2\pi}{L}\vec{n}} L^{-3} \Rightarrow \int \frac{d^3p}{(2\pi)^3} \quad \left(\text{sometimes written } \int d\!\!{}^-{}^3 p\right)$$

and such that $\tilde{f}(p) \to$ something finite in the $L \to \infty$ limit

Only way: change normalization I use for $a_p^\dagger$.

I wanna do that anyways, since $[a_p, a_q^\dagger] = \delta_{pq}$ Kronecker δ

Not limit of Dirac delta.

So I replace:

old normalization

$$a_{old,p} = \frac{1}{\sqrt{2\omega_p L^3}} \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \left(\omega_p \phi(x) + i\Pi(x)\right)$$

state: $|f\rangle = \sum_{p=\frac{2\pi}{L}\vec{n}} \tilde{f}_{old}(p)\, a^\dagger_{old,p} |0\rangle$

normalization: $\sum_p \tilde{f}^* \tilde{f}(p) = 1$

$$[a_p, a^\dagger_q] = \delta_{p,q}$$

Good: $\| a^\dagger |0\rangle \| = 1$ well normalized

Bad: to build wave packets, need $f(x) \propto L^{-3/2}$ funny.

new normalization

$$a_{new,p} = \frac{1}{\sqrt{2\omega_p}} \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \left(\omega_p \phi(x) + i\Pi(x)\right)$$

$$= L^{3/2} a_{old}$$

$|f\rangle = \sum_{p=\frac{2\pi}{L}\vec{n}} L^{-3} \tilde{f}_{new}(p)\, a^\dagger_{new,p} |0\rangle$

$\tilde{f}_{new}(p) = L^{3/2} \tilde{f}_{old}$, so

$$\sum_p L^{-3} \tilde{f}^*_{new} \tilde{f}(p) = 1 = \int \frac{d^3p}{(2\pi)^3} f^*_{new} f(p)$$

$$[a_p, a^\dagger_q] = L^3 \delta_{p,q}$$

$$\text{“} = (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \text{”}$$

Good: $L \to \infty$ limit now transparent.

Bad: $\| a^\dagger |0\rangle \| = L^3$ diverges

Must use wave packets to get physically reasonable states. (which is OK!)

Actually I will shift normalization even a bit more!

$a_{new,p} = L^{3/2} \sqrt{2\omega_p}\, a_{old}$. extra factor of $\sqrt{2\omega_p}$.

Then:

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left[a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$[a_p, a_{p'}^\dagger] = 2\omega_p (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')$$

so $\langle p | p' \rangle = 2\omega_p (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')$ if $|p'\rangle \equiv a_{p'}^\dagger |0\rangle$

$|f\rangle = \int \frac{d^3p}{(2\pi)^3 2\omega_p} f_p |p\rangle$ where $\int \frac{d^3p}{(2\pi)^3 2\omega_p} f^* f_p = 1$

correctly normalizes a wave packet.

why? $\underbrace{\int \frac{d^4p}{(2\pi)^4}}_{\text{obviously Lorentz inv}} \underbrace{(2\pi)\delta(p^2-m^2)}_{\text{Lorentz inv}} \underbrace{\Theta(p^0)}_{\text{inv. when next to }\delta\text{; explain}}$ is Lorentz invariant

$$= \int \frac{d^3p}{(2\pi)^3} \underbrace{\int \frac{dp^0}{\cancel{2\pi}} \cancel{2\pi}\, \delta((p^0)^2 - \omega_p^2)\, \Theta(p^0)}_{\frac{1}{2\omega_p}.\ \text{why?}}$$

So this normalization has smooth $L \to \infty$ limit & is naturally covariant.

Time evolution

Heisenberg picture $[H, a_p] = -\omega_p a_p$ easy to show

$$a_p(t) = e^{iHt} a_p e^{-iHt} = e^{-i\omega_p t} a_p$$

call $\omega_p = p^0$ or vice versa

nice, as $p^2 = m^2$ then, usual relativ. relation.

then $$\varphi(x,t) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left(e^{-ip_\mu x^\mu} a_p + e^{+ip_\mu x^\mu} a_p^\dagger \right)$$

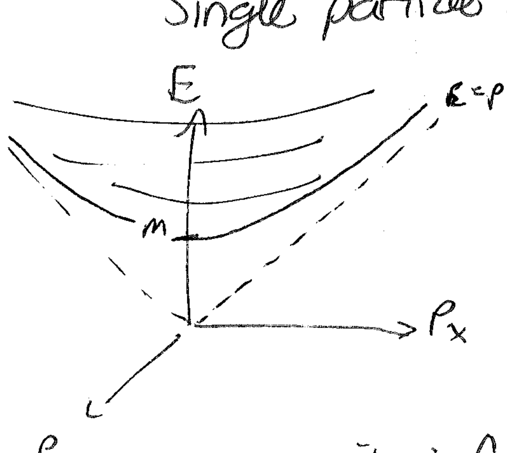
Oh look, using Heisenberg picture, $\varphi(x^\mu)$ has covariant expression

Heisenberg is way to go in QFT.

This thy is a thy of single particles & multi-part states.

Is there really an absolute difference? Yes!

Single particle: $E = \sqrt{p^2 + m^2}$. At rest, $E = m$

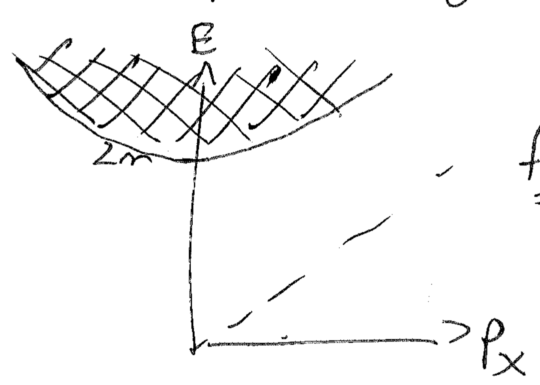


hyperbolic surface of revolution, "the mass shell" in E-p space.

2 particles: At Rest $E = \sqrt{\vec{p}_1^{\,2} + m^2} + \sqrt{\vec{p}_2^{\,2} + m^2}$ with $\vec{P}_1 = -\vec{P}_2$ (cm)
(cm frame)

But $|\vec{p}|$ can be anything. $\vec{p}^{\,2} \in [0, \infty]$ any value.

$E \geq 2m$ but can take any value $> 2m$.



fills the interior of a "mass shell"

physically different

We see this in 2-point correlation function

Define

$$G^{>}_{\varphi\varphi}(x) = \langle 0|\varphi(x)\,\varphi(0)|0\rangle \qquad \text{4-vector notation } x^\mu.$$

$$\tilde{G}^{>}_{\varphi\varphi}(q) \equiv \int d^4x\, e^{iq\cdot x}\, G^{>}_{\varphi\varphi}(x) \qquad \text{its momentum-space version}$$

Just compute it

$$G^{>}(x) = \langle 0| \int \frac{d^3p\, d^3k}{(2\pi)^3 2p^0 (2\pi)^3 2k^0} \left(a_p e^{-ip_\mu x^\mu} + a_p^\dagger e^{+ip_\mu x^\mu}\right)\left(a_k e^0 + a_k^\dagger e^0\right)|0\rangle$$

$$e^{-ip_\mu x^\mu}\, 2\omega_p (2\pi)^3 \delta^3(p-k)$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip_\mu x^\mu} \quad . \quad G^{>}(q) = 2\pi\,\delta(q^2-m^2)\,\Theta(q^0)$$

mass-shell requirement

$\varphi(0)$ makes a particle ($a_k^\dagger$)

It propagates in mass-shell respecting fashion

$\varphi(x)$ catches it & sees how it propagated to x. ~~Note, for $x^2<0$, $G^{>}(x)=0$ identically~~

Can I do this integral? yes!

for $x^2<0$: Use Lorentz, Luke. Go to frame $x^0=0$ $|\vec{x}| = -\sqrt{x^2} \equiv r$

Put it on z-axis.

$$G^{>}(r) = \int e^{ip_3 r} \frac{d^3p}{(2\pi)^3 2\sqrt{p^2+m^2}} = \frac{1}{16\pi^3}\int \frac{p^2 dp}{\sqrt{p^2+m^2}} \int_0^{2\pi} d\phi \underbrace{\int_{-1}^{1} d\cos\theta\, e^{ipr\cos\theta}}_{\frac{2}{pr}\sin pr}$$

$$= \frac{1}{4\pi^2 r}\int_0^\infty \frac{p\sin(pr)}{\sqrt{p^2+m^2}}dp$$

$$= \frac{m}{4\pi^2 r} K_1(mr) \quad \text{Modified Bessel Function} \longrightarrow \begin{cases} 1/4\pi^2 r^2 & mr \ll 1 \\ \dfrac{m^{1/2}}{2^{5/2}\pi^{3/2} r^{3/2}} e^{-mr} & mr \gg 1 \end{cases}$$

$$K_1(x) = \frac{\pi}{2} i e^{\frac{i\pi}{2}} H_1(ix)$$

$J_1 + iY_1$ Hänkel func.

Creepy part: $G^>(r) \longrightarrow {}^{1}/_{4\pi^2 r^2}$ at small r.

$\langle\varphi^2\rangle$ is like square of fluct. in fields.

But: $\langle\varphi^2\rangle = \lim_{r\to 0} \langle\varphi(r)\varphi(0)\rangle$

$= \lim_{r\to 0} \frac{1}{4\pi^2 r^2}$ eek!

At scale r, fluct. in field have size $1/2\pi r$

The $1/2\pi$ is nice – effects of fluct. will often have $(1/2\pi)^2$

The $1/r$ means field picture really weird.

Don't Ask "what is fluct. in φ at a point?"

Do ask "fluct at some scale r?"

say, smear field $\varphi_r \equiv \int \frac{d^3x}{(2\pi)^{3/2} r^3} e^{-x^2/2r^2} \varphi(x)$ (so normalized to 1)

$$\langle\varphi_r^2\rangle = \int \frac{d^3x\, d^3y}{(2\pi)^3 r^6} e^{-\frac{(x^2+y^2)}{2r^2}} \frac{1}{4\pi^2 (x-y)^2}$$

$$= \frac{1}{8\pi^2 r^2}$$ yup. Gets bigger the smaller you make r.

Oh, and $G^>(t)$ = analytic continuation of $G^>(r)$

$= \frac{m}{8\pi t} H_1(t)$ Hänkel function