Now turn on interactions

$$\mathcal{L} = \oint \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{24}\varphi^4$$

What do I know about this theory?

1 - Is it Harmonic Oscillators?
2 - Are there particles?
3 - Does it exist at all?

Totally not clear!
1 - NO
2 - Yes, but see 3
3 - Yes for finite UV cutoff but limit $a \to 0$ ($\Lambda \to \infty$) is subtle - theory "loses" $\lambda\varphi^4$ term

How can I learn anything about such a theory?
What if I could somehow compute $\langle 0|\varphi(x)\varphi(0)|0\rangle = G^>(x)$.
What would it teach me? (Free thy - know about particles)

Or more generally, for some local operator $\mathcal{O}$ in QFT, what do we learn from

$$G^>_{\mathcal{O}\mathcal{O}}(x) = \langle 0|\,\mathcal{O}^\dagger(x)\,\mathcal{O}(0)|0\rangle \quad ?$$

Well, assume there are states (maybe contin or discrete..)
organize as e-states of P^μ ($P^\mu = \int T^{\mu 0} d^3x$) $|n\rangle$: $\hat{P}^\mu|n\rangle = p_n^\mu|n\rangle$

Assume $P_n^0 > 0$ except for vacuum ("Mass Gap")

$$\mathcal{O}(0)|0\rangle = \overbrace{\sum_n}^{\text{complete set of states}} |n\rangle\underbrace{\langle n|\mathcal{O}(0)|0\rangle}_{C_{\mathcal{O},n} \text{ some c number}} = \sum_n C_{\mathcal{O}n}|n\rangle$$

$$G^>_{\mathcal{O}\mathcal{O}}(x) = \langle 0|e^{i\hat{P}^\mu x_\mu}\mathcal{O}^\dagger e^{-i\hat{P}^\mu x_\mu}\mathcal{O}|0\rangle$$

$$= \sum_{nm} C^*_{\mathcal{O}m}C_{\mathcal{O}n}\langle m|\underbrace{e^{-i\hat{P}^\mu x_\mu}|n\rangle}\, e^{-ip_n^\mu x_\mu}\langle m|n\rangle = \sum_n e^{-ip_n^\mu x_\mu} C^*_n C_n$$

If $\langle 0|\mathcal{O}|0\rangle = C_0 \neq 0$, Replace $\mathcal{O} \to \mathcal{O} - C_0$

(vacuum $\to$ vacuum is boring)

Then $P_n^0 > 0$

Note: $$1 = \int_{M^2}^{\infty} \frac{ds}{2\pi} \int \frac{d^4 p}{(2\pi)^4} (2\pi)\, \delta(p^2 - s)\, \Theta(p^0)\, (2\pi)^4\, \delta^4(p - P_n)$$

trivially

$$G^{>}_{\mathcal{O}\mathcal{O}}(x) = \int_{M^2}^{\infty} \frac{ds}{2\pi} \int \frac{d^4 p}{(2\pi)^4}\, e^{-ip_\mu x^\mu}\, \Theta(p^0)\, 2\pi\, \delta(p^2 - s)$$
$$\times \Big[\underbrace{\sum_n (2\pi)^4 \delta^4(p - P_n)\, C_n^* C_n}_{} \Big]$$

this is unchanged by boosts: value for P_n^μ = value for $\Lambda^\mu{}_\nu P_n^\nu$

Depends only on P_n^μ through invariant P_n^2.

Name it $\rho(P_n^2) = \rho(s)$

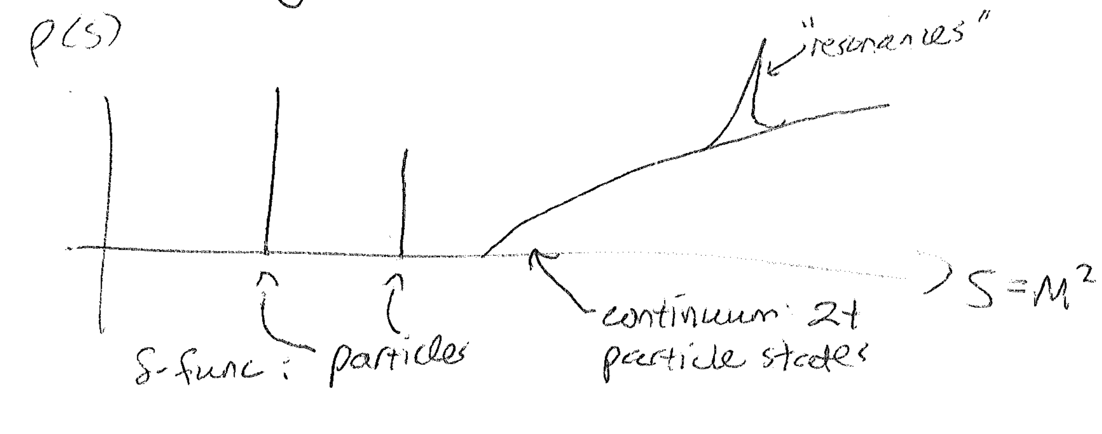
$$G^{>}_{\mathcal{O}\mathcal{O}}(x) = \int \frac{ds}{2\pi}\, \rho_{\mathcal{O}\mathcal{O}}(s)\, \Delta(x, s)$$

$\hookrightarrow \int \frac{d^4 p}{(2\pi)^4} (2\pi)\, \delta(p^2 - s)\, \Theta(p^0)\, e^{-ip\cdot x}$

Behavior we saw for a massive free particle.

Integral over m^2 values, of

$\rho(s) \times$ what a massive particle does.



What else is in correlation functions?

$G^<(x) = \langle 0| \varphi(0)\varphi(x)|0\rangle$ = same as $G^>$ but $(-p^0)$

$\sigma(x) = G^>(x) - G^<(x) = \langle 0| [\varphi(x),\varphi(0)]|0\rangle$

Cares about causality & teaches about propagation (how?)

First: for $x^2<0$, $[\varphi(x),\varphi(0)]=0$ by causality. Also explicitly

For instance: $\mathcal{L} \to \mathcal{L}_{old} - \varphi(x)J(x)$

$J(x)$ "external" (I-choose) x^μ-dependent function. "Handle" to "wiggle" field.

With this $\mathcal{L}$, what is $\langle 0, t=\infty| \varphi(y)|0,t=-\infty\rangle$?

Insert time evolution $\langle 0| \mathcal{U}^\dagger(y^0,-\infty)\,\varphi(\vec{y})\,\mathcal{U}(y^0,-\infty)|0\rangle$

where $\mathcal{U}$ obeys $H\mathcal{U}(t-t_0) = i\partial_t \mathcal{U}(t-t_0)$ is time-evolution operator.

$\mathcal{L} = \mathcal{L}_0 + \underbrace{\mathcal{L}_1}_{\to -\varphi(x)J(x)}$ expand to 1-order in $\mathcal{H}_1$

$\mathcal{H} = \mathcal{H}_0 + \underbrace{\mathcal{H}_1}_{\to +\varphi(x)J(x)}$

$\mathcal{U}(t,-\infty) = 1 - i\int d^3x \int_{-\infty}^{t} \varphi(x,t')\,J(x,t')\,dt'$ in int. picture (Time-Dependent Pert. Thy)

$$\langle 0|\varphi(y)|0\rangle = \langle 0|\left(1+i\int_{x^0<t} d^4x\,\varphi(x)J(x)\right)\varphi(y)\left(1-i\int_{z^0<t} d^4z\,\varphi(z)J(z)\right)|0\rangle$$

$$= -i\int_{x^0<t} d^4x\,\langle 0|[\varphi(y),\varphi(x)]|0\rangle\,J(x)$$

$$= -i\int_{x^0<t} d^4x\,J(x)\,\sigma(y-x)$$

$\sigma(y-x)$ tells how cause at x affects field at y.

Technically the thing we wanted there was:

$$G_R(y-x) = \langle 0|\,[\varphi(y),\varphi(x)]\,|0\rangle\, \Theta(y^0-x^0)$$

$$\langle \delta\varphi(y)|0\rangle_J = \int d^4x\ -iJ(x)\, G_R(y-x) \quad \text{Retarded Function}$$

~~Usually~~

~~In spacetime,~~

Usually we absorb the i factor into definition of G_R:

$$G_R(y-x) = -i\,\langle 0|\,[\varphi(y),\varphi(x)]\,|0\rangle\,\Theta(y^0-x^0)$$

Note: I can define $G_R(p)$ at positive-imaginary-part-frequency: $G_R(p) = \int d^4y\, e^{ip_\mu y^\mu} (-i)\langle 0|\,[\varphi(y),\varphi(0)]\,|0\rangle\,\Theta(y^0)$

for $p^0 = p^0_r + i p^0_i$, that's $\int d^4y\ e^{-p^0_i y^0}\, e^{i y^\mu \mathrm{Re}\, p_\mu}$

decaying factor. Only evaluate at $y^0>0$!

$G_R(p)$ is defined & free of singularities in upper-half p^0 plane.

Let's compute it in our free theory!

We saw that $\langle 0|\,\varphi(y)\,\varphi(0)\,|0\rangle = \int \frac{d^4q}{(2\pi)^4}\, e^{-iq_\mu y^\mu}\ \Theta(q^0)\, 2\pi\,\delta(q^2-m^2)$

$\langle 0|\,\varphi(0)\,\varphi(y)\,|0\rangle =$ " " $\Theta(-q^0)$

$$G_R(p) = \int d^4y\ \Theta(y^0)\, e^{ip_\mu y^\mu} \int \frac{d^4q}{(2\pi)^4}\left[-i2\pi\,\delta(q^2-m^2)\left(\Theta(q^0)-\Theta(-q^0)\right)\right] e^{-iq_\mu y^\mu}$$

the $\int d^3y\, e^{-i(\vec{p}-\vec{q})\cdot\vec{y}} \to (2\pi)^3 \delta^3(\vec{p}-\vec{q})$

performs the $\int \frac{d^3q}{(2\pi)^3}$ part, leaving

$$G_R(p) = \int dy^0\, \Theta(y^0) \int \frac{dq^0}{2\pi} (-2\pi i)\, e^{i(p^0-q^0)y^0} \left(\Theta(q^0) - \Theta(-q^0)\right) \cdot \delta(q_0^2 - \omega_p^2)$$

How do I do that?

1) do $\int dq^0$ using the δ-function. $q^0 = \omega_p$ with $+$
$q^0 = -\omega_p$ with $-$

2) again take $p^0 \to p^0 + i\frac{\varepsilon}{2}$ ← we will see why
so I can do y-int.

q-int: $\frac{-i}{2\omega_p}\left(e^{i(p^0-\omega_p)y^0} - e^{i(p^0+\omega_p)y^0}\right)$

Recall $\int_0^\infty dy\, e^{-ny} = \frac{1}{n}$. $\int dy\, e^{-(\varepsilon - ik)y} = \frac{1}{\varepsilon - ik}$. so

$$\left(\frac{-i}{2\omega_p}\right)\int_0^\infty dy^0 \left(e^{i(p^0-\omega_p)y} - e^{-(p^0+\omega_p)y}\right) e^{-\varepsilon/2\, y_0} = \left(\frac{-i}{2\omega_p}\right)\left(\frac{1}{\frac{\varepsilon}{2} - i(p^0-\omega_p)} - \frac{1}{\frac{\varepsilon}{2} - i(p^0+\omega_p)}\right)$$

$$= \frac{1}{2\omega_p} \frac{2\omega_p}{(p^0-\omega_p+\frac{i\varepsilon}{2})(p^0+\omega_p+\frac{i\varepsilon}{2})} = \frac{1}{p_0^2 - \omega_p^2 + i\varepsilon p_0} = \frac{1}{p^2 - m^2 + i\varepsilon p_0}$$

Some problems call for

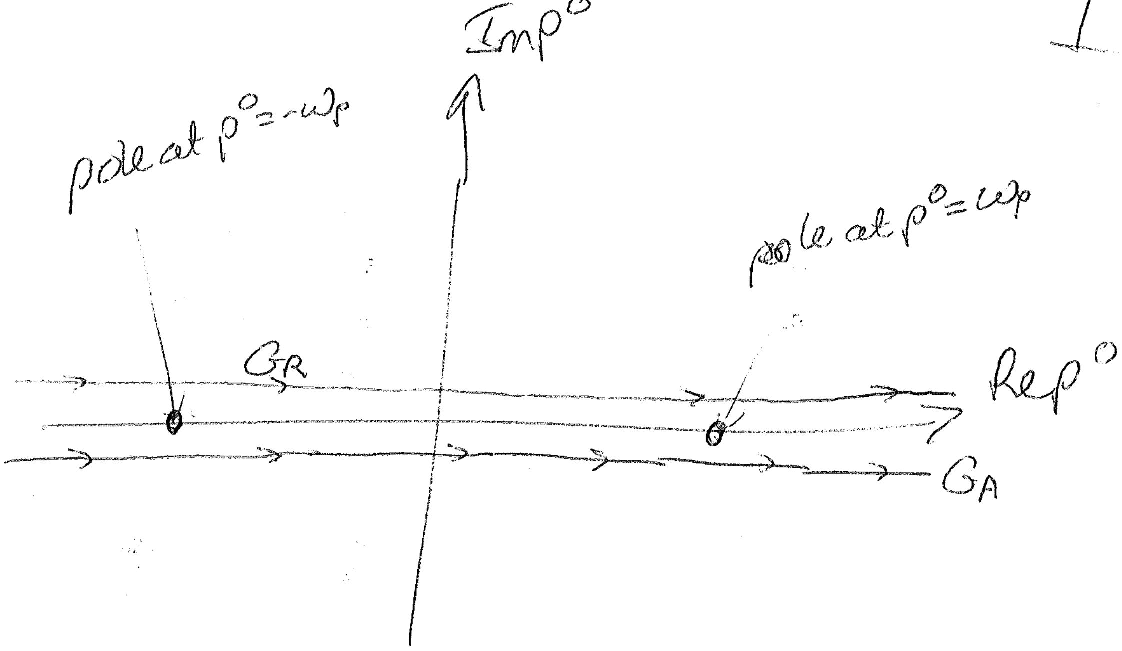
$G_A(x) = \underset{\text{note}}{(+i\langle[\varphi(x),\varphi(0)]\rangle)}\ \Theta(-x^0) \to G_A(p) = \frac{1}{p^2 - m^2 - i\varepsilon p_0}$ sign on ε tells side of p^0 = Real contour to use.

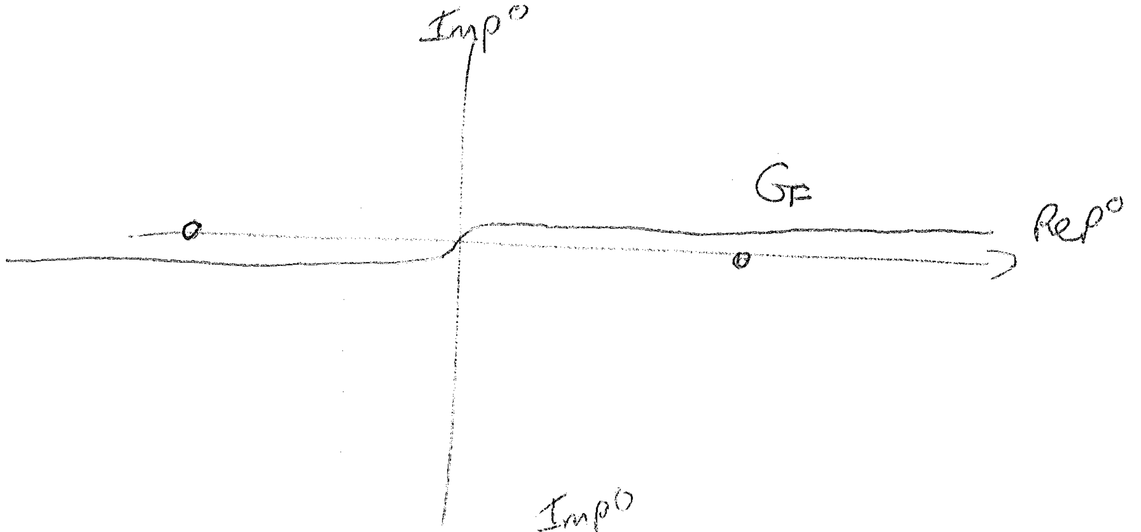
Some problems need

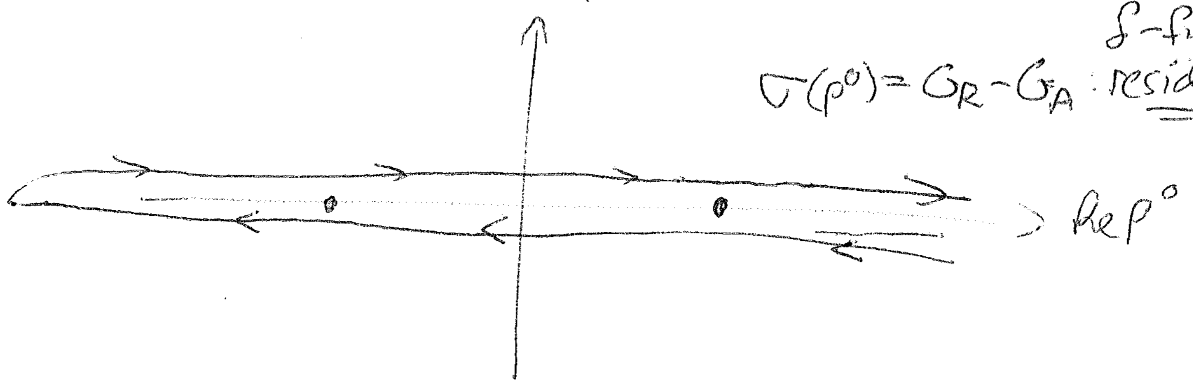
$$G_F(x) = -i\langle 0|\left(\varphi(x)\varphi(0)\Theta(x^0) + \varphi(0)\varphi(x)\Theta(-x^0)\right)|0\rangle = -i\langle 0|T(\varphi(x),\varphi(0))|0\rangle$$

$$G_F(p) = \frac{1}{p^2 - m^2 + i\varepsilon}$$. Book uses Δ_F

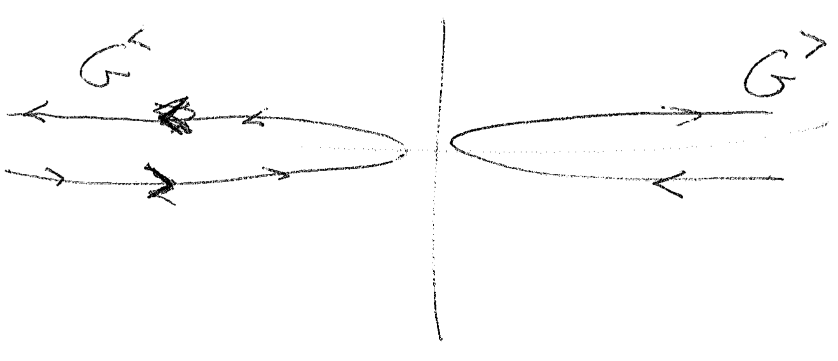
LGPG







$\sigma(p^0) = G_R - G_A$: δ-func. times residues of poles



feels singularities but only on ½ the axis.