In state: $\left[\prod_{i=1}^{m} \int d^4y_i \; \frac{\psi_{p_i}(y_i - x_i)}{\sqrt{Z}} \varphi(y_i)\right]|0\rangle$

well separated particles. x_i early times, spacelike separated

Out state: $\left[\prod_{j=1}^{n} \quad \cdots \quad \right]|0\rangle \qquad x_i$ late times.

Evolution of one to other: $S = \langle \text{out} | \text{in} \rangle$ will "in" evolve into "out"?

$$\langle 0| \left[\prod_{j=1}^{n} \int d^4z_j \; \frac{\psi^*(z_j - x_{f,j})}{\sqrt{Z}} \varphi^*(z_j)\right]\left[\prod_{i}^{m} \int d^4y_i \frac{\psi(y_i - x_{i,i})}{\sqrt{Z}} \varphi(y_i)\right]|0\rangle$$

$$= \underbrace{\frac{1}{Z^{\frac{m+n}{2}}} \int d^4z_1 \ldots d^4z_n \, d^4y_1 \ldots d^4y_m \, \psi^*(z_j - x_{f_j}) \, \psi(y_i - x_{i,i})}_{\text{state preparation}} \; \underbrace{\langle 0| \varphi^*(z_1) \ldots \varphi^*(z_n) \varphi(y_1) \ldots \varphi(y_m)|0\rangle}_{\text{correlation function of my thy}}$$

All $\varphi^*(z)$ come after all $\varphi(y)$ in operator ordering

The $\varphi^*(z)$ are spacelike separated - exact order irrelevant

the $\varphi(y)$ " " " " " "

So just T-order the whole thing!

The part the theory brings to the table is

$$G_{\varphi_1^* \ldots \varphi_n^* \varphi_1 \ldots \varphi_m}(z_1 \ldots z_n; y_1 \ldots y_m) = \langle 0| T(\varphi^*(z_1) \ldots \varphi^*(z_n) \varphi(y_1) \ldots \varphi(y_m))|0\rangle$$

All we could want to know is in T-ordered vac-vac correlation functions like this.

But it's actually not quite $G_{\varphi_1^* \ldots}$ that we are after

LSZ Reduction Formula

What do all these <u>wave packets</u> have to do with things? Can I get rid of them? Especially of the $\int dt\, f(t) e^{\pm iEt}$?

Yes. Think about one initial particle:

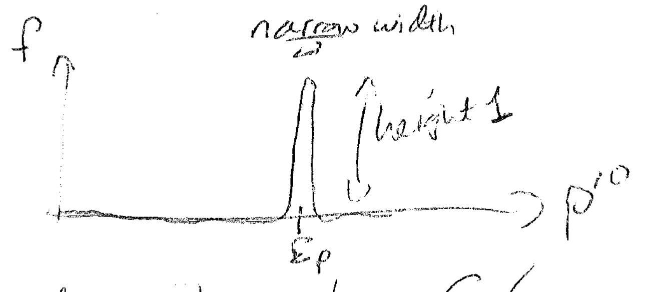
$$\underbrace{\int d^3y \int dy^0\, f(y^0-x^0) e^{-iE_p(y^0-x^0)}\, \varphi(\vec{y}-\vec{x})\, G(\ldots, y, \ldots)}$$

same as $\int d^3p'\, \tilde{\varphi}(p')\, G(\ldots, p', \ldots)$

And $\int dy^0 f(y^0) e^{-iE_p y_0} G(\ldots, y^0, \ldots)$

$$= \int dp'^0\, \tilde{f}(p'^0 - E_p)\, G(\ldots, p'^0, \ldots)$$

But



Not a δ function! Only height 1

Integral <u>vanishes</u> <u>unless</u> $G(\ldots, p'^0, \ldots)$ has a <u>pole</u>

$$G(\ldots, p^\mu, \ldots) \equiv \frac{1}{(p_0^2 - E_p^2 + i\epsilon)}\, G_{Amp}(\ldots p^\mu, \ldots)$$

~~really it was $f(y^0)e$~~ Residue of pole <u>only</u> is captured by integral. (Subtleties here - we skip for now). Residue: $\frac{1}{2E}$

Amplitude $\langle k_1 \ldots k_n\, out | p_1 \ldots p_m\, in \rangle = \int d^3z_1 \ldots d^3z_n\, d^3y_1 \ldots d^3y_m\, \frac{\varphi^*_{k_1}(z_1)}{2E} \ldots \frac{\varphi_{p_1}(y_1)}{2E} \ldots$

$$\times\, G^{AMP}(z_1, \ldots, y_1, \ldots)$$

Consider $G^{AMP}(k_1 \ldots k_n; p_1 \ldots p_m)$ – what you really compute and really use.

~~Techn~~

What does it have to do with actual scattering?

I really want Probability $|\langle out | in \rangle|^2$. Or do I?

I will never see that a final state is some specific wave packet.

Xpt limitations: at most, see that final particle in some ~~~~ k-window.

$$\langle out | \sim \int d^3z \, \psi_k(z) \langle 0 | \varphi(z)$$

$$|out\rangle\langle out| \sim \int d^3z_1 d^3z_2 \, \psi^*(z_1)\psi(z_2) \; \varphi|0\rangle\langle 0|\varphi \quad \to G(\ldots)$$

I want $\sum\limits_{\substack{\text{all independent}\\ \text{wave packets which}\\ \text{have } k \in \text{ some range}}}$ " How do I do that?

It's the projection operator $\displaystyle\int_{[range]} \frac{d^3\vec{k}}{(2\pi)^3 2k^0} \left(G^*_{Amp} G_{Amp}(k, \ldots \quad) \right)$

Sum over all states corresp. to some momentum range k is $\displaystyle\int_{[range]} |k\rangle\langle k|$ which gives $|G_{Amp}|^2$ in that k range.

Proof: $\displaystyle\int \frac{d^3k}{(2\pi)^3 2k^0} |k\rangle\langle k|$ is a projection operator with normalization 1.

what about initial state?

I can write $\int d^3x_1\, d^3x_2 \; \dfrac{\psi_p^*(x_1)}{2E} \dfrac{\psi_p^*(x_2)}{2E} \; G^*(\ldots x_1, \ldots) \; G(\ldots, x_2, \ldots)$

$$= \int \frac{d^3p_1\, d^3p_2}{(2\pi)^6\, 2p_1^0\, 2p_2^0} \; \psi_p^*(p_1)\, \psi_p(p_2) \; G^*(\ldots, p_1, \ldots) \; G(\ldots, p_2, \ldots)$$

Specialize to 2 incoming particles.

Answer depends on specifics of wave packets! As it should!

Normally one is interested in an $\int$ over all final states in some range of momentum, of the probability of a scattering.

For given initial wave packets $\psi_1, \psi_2 \ldots$
this corresponds to

$$\prod_f \int_{\text{some range}} \frac{d^3 p_{f_i}}{(2\pi)^3 2E_f} \int \prod_i \frac{d^3 p_i \, d^3 p_i'}{(2\pi)^3 2E_i \, (2\pi)^3 2E_i} \prod_i \psi_i^*(p_i')\psi_i(p_i) \; G^*_{Amp}(p_i', p_f) \, G_{Amp}(p_i, p_f)$$

Here I have chosen the (improperly normalized) final wave ~~pa~~ functions $|p_f\rangle$ and fixed normalization by $\int dp_f$:

$$\int_{\text{range of mom}} \frac{d^3 p_f}{(2\pi)^3 2p_f} |p_f\rangle\langle p_f| \quad \text{is a properly normalized projection operator.}$$

I also implicitly put right $Z^{-(n+m)}$ into $G^*_{Amp} G_{Amp}$

Given the wave packets and $|G_{Amp}|^2$, this gives scattering probability.
However it depends in detail on wave packets ψ_i (as it should).

It would be nice to define universal quantities that give the physics — then you $\times$ by your beam characteristics to get scatt. rates.

P2

$L\cancel{3}$ 10

Very hard to get 3-body scatt. to occur - usually only in dense environments. Return to later.

2-body scattering: ~~typical problem~~ has two momenta P_1, P_2

Center-of-mass frame: where $P = P_1 + P_2$ is purely temporal.

(note, $(P_1+P_2)^2 = P_1^2 + 2P_1 \circ P_2 + P_2^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2\vec{P}_1 \circ \vec{P}_2 \equiv$ "s"

$E \geq |\vec{p}|$ and $|\vec{p}_1 \circ \vec{p}_2| \leq |p_1||p_2|$ so this is ≥ 0.
Only $= 0$ if $P_1 P_2$ both lightlike & colinear (parallel and same-speed)

"Beam frame" any frame where $\vec{P}_1, \vec{P}_2$ are $\parallel$ or anti$\parallel$. Let's work in such a frame.

"Lab" frame: where one $\vec{p} = 0$ (one particle at rest)

Specialize for sake of argument to separable wave func.

$$\psi(\vec{x}) = \psi_\perp(x_\perp)\psi_\parallel(x_\parallel).$$

$$\int \frac{d^2p_\perp}{(2\pi)^2} \psi_\perp^* \psi_\perp = 1, \quad \int \frac{dp_\parallel}{2\pi\, 2E} \psi_\parallel^* \psi_\parallel = 1$$

$\psi_\perp(p_\perp)$ centered around $p_\perp = 0$
$\psi_\parallel(p_\parallel)$ centered around $p_\parallel = |p|$

$$\text{Probability } dP = \prod \int \frac{d^3 p_{f,n}}{(2\pi)^3 2p_{f,n}} \int \frac{d^2p_{1\perp}\, d^2p_{2\perp}}{(2\pi)^4} \frac{d^2p'_{1\perp}\, d^2p'_{2\perp}}{(2\pi)^4} \frac{dp_{1\parallel}\, dp_{2\parallel}\, dp'_{1\parallel}\, dp'_{2\parallel}}{(2\pi)^4\, 16 E_1^2 E_2^2}$$

$$\times \psi_{1\perp}^*(P'_{1\perp})\psi_{1\perp}(P_{1\perp})\psi_{2\perp}^*(P'_{2\perp})\psi_{2\perp}(P_{2\perp}) \quad (2\pi)^4\delta^4(P_1+P_2-\Sigma P_f)\,(2\pi)^4\delta^4(P'_1+P'_2-\Sigma P_f) \times |M|^2$$

same $\parallel$

P3

L13/10

Try to do p, p' $\int$'s at fixed P_f but remembering a range is $\int$ over.

$|M|^2$ is smooth function of p's $\rightarrow$ use central values of wave packets and move outside.

Use $(2\pi)^4 \delta^4(P_1+P_2-\Sigma P_f)\,(2\pi)^4 \delta^4(P_1'+P_2'-\Sigma P_f)$

$$= (2\pi)^4 \delta^4(P_1+P_2-\Sigma P_f)\,(2\pi)^4 \delta^4(P_1+P_2-P_1'-P_2').$$

Move $\delta^4(P_1+P_2-\Sigma P_f)$ outside on replacing $P_1 P_2 \to$ central value. Valid if $\int$ range in P_f's wider than width of packets in p space.

Use $(2\pi)^2 \delta^2(P_{1\perp}+P_{2\perp}-P_{1\perp}'-P_{2\perp}')$ to do $P_{2\perp}'$ integral. remaining $\int$'s in $\perp$ space

$$\int_{P_{1\perp} P_{2\perp} P_{1\perp}'} \psi_\perp^*(P_1')\,\psi_\perp(P_1)\,\psi_\perp^*(P_1+P_2-P_1')\,\psi_\perp(P_2)$$

$$\psi(p) = \int d^2x\, e^{+i\vec{p}\cdot\vec{x}}\,\psi(x)$$

$$\int_{x_1 x_2 x_1' x_2'} \underbrace{\int_{P_1 P_2 P_1'} e^{+i[P_1\cdot(x_1-x_2')+P_2\cdot(x_2-x_2')+P_1'\cdot(x_2'-x_1')]}}_{\delta(x_1-x_2')\delta(x_2-x_2')\delta(x_2'-x_1')} \psi^*(x_1')\psi(x_1)\psi^*(x_2')\psi(x_2)$$

$$\rightarrow \int d^2x\, \underbrace{\psi_\perp^*\psi_\perp(x)}_{\text{part 1}}\, \underbrace{\psi_\perp^*\psi_\perp(x)}_{\text{part 2}}$$

Integral of overlap of densities in plane.

For particles on-axis, of extent $L_0 \times L_0$, $\psi^*\psi \sim \frac{1}{L^2}$ so this $\rightsquigarrow 1/L^2$.

Expected thing so particles which miss each other don't scatter, "spread out" scatter less than "focused" etc.

Thing multiplying this, to get a probability, has units of area and is the key thing to learn. Defined to be the "cross-section" because Prob to scatt is same as if one particle were a point & other had area $\square$ = cross section "σ"

[Technically we have set up calc. of differential σ,

$$d\sigma = \prod_{p_f} \frac{d^3p_f}{(2\pi)^3 2p_f} \times (\ldots)$$]

Continuing : ~~Move $\delta^2(p_1'+p_2'=p_f)$ outside to force~~

$$\int \frac{dp_{1z}'\,dp_{1z}''\,dp_{2z}'\,dp_{2z}''}{(2\pi)^4 (4E_1E_2)^2} (2\pi)^2\, \delta(p_{1z}'+p_{2z}'-p_{1z}''-p_{2z}'')\, \delta(p_1^{0\prime\prime}+p_2^{0\prime\prime}-\cancel{p_f^0})\ \varphi^*_{(p_1'')}\varphi_{(p_1')}\varphi^*_{(p_2'')}\varphi_{(p_2')}$$

$p_1^{\prime 0}-p_2^{\prime 0}$

Use δ func's to do momentum $\int$'s over p'''s :

$$\int dp_{2z}''\ \delta(p_{z1}'+p_{z2}'-p_{z1}''-p_{z2}'') = 1$$ ~~and $p_2'' \to p_1'+p_2'-p_1''$~~

$$p_{2z}'' \to p_{1z}'+p_{2z}'-p_{1z}''$$

Note p_i^0 dependent variable meaning $\sqrt{m^2+p^2}$

Now, $p_2^{\prime\prime 0} \equiv \sqrt{m^2+(p_1'+p_2'-p_1'')^2}$ (!)

So doing $\int dp_{1z}''\ \delta\left[p_1^{0\prime}+p_2^{0\prime}-p_1^{0\prime\prime}-p_2^{0\prime\prime}\right]$ gets Jacobian

$$\to 1\Big/\left(\frac{d}{dp_{1z}''}(p_1^{0\prime\prime}+p_2^{0\prime\prime})\right) = \frac{1}{\left(\frac{d(\sqrt{p^2+m^2})}{dp}+\frac{d(\sqrt{m^2+(p_1+p_2-p)^2})}{dp} = V_1 \mp V_2\right)}$$

$$= \frac{1}{V_1-V_2}$$. Head to head: $V_1-V_2 = 2$ at ultra-relativistic speeds

This leaves

$$\frac{1}{V_1-V_2}\int \frac{dp_{1z}\,dp_{2z}}{(2\pi)^2(4E_1E_2)^2}\ \varphi^*\varphi(p_1)\,\varphi^*\varphi(p_2) = \frac{1}{2E_1 2E_2(V_1-V_2)}$$

(normalization of φ's)

p5

LB 10

So at last,

$$\sigma = \left(\prod_{p_f} \int \frac{d^3 p_f}{(2\pi)^3 2p_f^0} \right) \frac{1}{4E_1 E_2 (V_1 - V_2)} |M|^2 (2\pi)^4 \delta^4(p_1 + p_2 - \Sigma p_f)$$

Note: only 1 δ^4 even though 2 initially - one used up dealing with wave packets.

$\int \frac{d^3 p_f}{(2\pi)^3 2p_f^0}$ is Lorentz invariant: $\int \frac{d^4 p_f}{(2\pi)^4} \delta(p_f^2 - m^2)\theta(p_f^0)$ as we saw

$\frac{1}{4E_1 E_2 |V_1 - V_2|}$ Also Lorentz invariant!

General Statement

If $\frac{\#\text{ particles}}{\text{space volume} * \text{mom space region}}$: $n = \int d^3x \int \frac{d^3p}{(2\pi)^3 2p^0} f(x,p)$

Then $\frac{\#\text{ scatterings}}{\text{space-time volume}} = \prod_{p_f} \int \frac{d^3 p_f}{(2\pi)^3 2p_f^0} \prod_{p_i} \int \frac{d^3 p_i}{(2\pi)^3 2p_i^0} \prod_i f(p_i) |M|^2 (2\pi)^4 \delta^4(\Sigma p_i - \Sigma p_f)$

Missing $\frac{1}{|V_1 - V_2|}$ which is flux factor: