Path Integral

We want to know all quantities of form

$$\langle 0 | T\left(\varphi_1(x_1)\, \varphi_2(x_2) \dots \varphi_n(x_n) \right) | 0 \rangle$$

Step 1 - modify $\mathcal{L} = \frac{\partial_\mu \varphi_a \partial^\mu \varphi_a}{2} - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{24}\varphi^4 \underbrace{- J_a(x)\,\phi_a}_{\text{new.}}$

$$H = \int d^3x \left(\frac{\pi^2}{2} + \frac{|\nabla \varphi|^2}{2} + \frac{m^2 \varphi^2}{2} + \frac{\lambda \varphi^4}{4!} + J\phi \right)$$

time-evol: $e^{-i\int_{t_i}^{t_f} H\,dt}$, interaction picture $T\exp -i\int J(x)\varphi(x)\,d^4x$

$$\langle 0 | T(\varphi_1(x_1) \dots \varphi_n(x_n)) | 0 \rangle = \frac{i^n\, \delta^n}{\delta J_1(x_1)\, \delta J_2(x_2) \dots} \langle 0 | \;\; | 0 \rangle \Big|_{J=0}$$

Apply derivatives: $\frac{\delta}{\delta J(x_1)} \int d^4y\, J(y)\varphi(y) = \int d^4y\, \delta^4(x_1 - y)\varphi(y) = \varphi(x_1)$

J-derivatives "pull out" $\varphi_i(x_i)$ factors. Then set $J=0$

(or, take n'th term in Taylor expansion in J)

So let's name $Z(J) = \langle 0 | e^{-i\int H dt} | 0 \rangle$ with $H = H_0 + \int J\varphi\, d^3x \dots$

If I could find $Z(J)$, I would know all correl. func's from its derivs. $Z(J)$: generating Func. of all Correlation Functions

How do I get $Z(J)$??

Path Integral

Consider Quantum Mechanics w. canonical coord. Q_a, conjugate momenta P_a. Assume

$$H = \underbrace{\frac{1}{2}\sum_a P_a^2}_{} + V(Q_a)\ , \quad V \text{ some function of } Q\text{'s}.$$

(or $\frac{1}{2}M_{ab}P_aP_b$)

The Q_a could be $\varphi(\vec{x})$ in my discrete-space, finite-space "regularized" version of QFT. Again, I must regulate from the start.

Most general thing I want: $\langle \psi_f | \mathcal{U}(t_f - t_i) | \psi_i \rangle$

$$|\psi_{f/i}\rangle = \underbrace{\int dQ_1 \ldots dQ_n}_{\int dQ_i \text{ for short}} \psi_{f/i}(Q_i)\, |Q_i\rangle$$

coord-space basis for states

$$\mathbb{1} = \int dQ_i\, |Q_i\rangle\langle Q_i|$$ resolution of identity // completeness of states

Write $\mathcal{U}(t_f - t_i) = \cancel{\mathcal{U}(t_f - t_N)}\,\mathcal{U}(t_N - t_{N-1}) \ldots \mathcal{U}(t_2 - t_1)\mathcal{U}(t_1 - t_i)$

where I introduce $N \gg 1$ intermediate times, $t_N - t_{N-1} = \frac{t_f - t_i}{N}$ $\begin{pmatrix} t_f = t_N \\ t_i = t_0 \end{pmatrix}$

$$Z(J) = \langle 0 |\, \mathcal{U}(t_n - t_{n-1})\, \mathcal{U}(t_{n-1} - t_{n-2}) \ \ldots \ \mathcal{U}(t_1 - t_0)\, | 0 \rangle$$

Insert: $\int dQ_{i,N} |Q_{i,N}\rangle\langle Q_{i,N}|$ $\quad \int dQ_{i,N-1} |Q_{i,N-1}\rangle\langle Q_{i,N-1}|$ $\quad \int dQ_{i,0} |Q_{i,0}\rangle\langle Q_{i,0}|$

How many integrals? $(N+1) \times$ number of generalized coord.

I want $N \to \infty$, $\frac{t_f - t_i}{N} \to 0$ limit. Limit of many $\int$'s.

(don't forget, one Q_i per point in space!!)

L9P3

One integral { per field, per point in space, per point in time } point in spacetime. Aha!! Space, time on same footing!

$$\int \underbrace{dQ_N \, dQ_{N-1} \ldots dQ_0}_{\substack{(N+1) * \#\text{pts in space} \\ * \#\text{fields of } \mathcal{S}\text{'s}}} \; \langle 0 | Q_N \rangle \langle Q_N | \mathcal{U}(\tfrac{\Delta t}{N}) | Q_{N-1} \rangle \ldots$$

or ψ_f

$$\ldots \langle Q_2 | \mathcal{U}(\tfrac{\Delta t}{N}) | Q_1 \rangle \langle Q_1 | \mathcal{U}(\tfrac{\Delta t}{N}) | Q_0 \rangle \langle Q_0 | \psi_i \rangle$$

or 0

Name this $\int \mathcal{D}Q$

I have $\langle Q_M | \mathcal{U}(\frac{\Delta t}{N}) | Q_{M-1} \rangle$ appearing N times.

Can expand in $N \gg 1$, $\Delta t / N$ small. But must work to 1st order in $1/N$. $1/N^2$ I can drop.

actually, leave as exponential $e^{\frac{-iMPP\Delta t}{2N}} (1-i)$

$- i \frac{\Delta t}{N} V(Q)$

$$\langle Q_M | (1 - i \frac{\Delta t}{N} M_{ab} \frac{P_a P_b}{2}) | Q_{M-1} \rangle$$

How do I evaluate that? Use momentum basis

$1 = \int \frac{dP_M}{2\pi} | P_M \rangle \langle P_M |$ insert here

$\hat{P}_a \to P_{aM}$ on state. And $\langle Q_M | P_M \rangle = e^{-iQ_{aM} P_{aM}}$

so I get $\int \frac{dP_M}{2\pi} e^{iP_{aM}(Q_{M-1} - Q_M)} e^{-i \frac{\Delta t}{2N} M_{ab} P_{aM} P_{bM}}$

How do I do that integral?

Complete the square

$$\exp\Big[-\frac{i\Delta t}{N} M_{ab}\frac{P_{am}P_{bm}}{2} + i P_{am}\left(Q_{am-1} - Q_{am}\right) - i\frac{N}{\Delta t}\left(Q_{m-1}-Q_m\right)_a \frac{M^{-1}_{ab}}{2}\left(Q_{m-1b}-Q_{mb}\right)$$

$$+ i\frac{N}{\Delta t}\left(Q_{m-1a}-Q_{ma}\right)\frac{M^{-1}_{ab}}{2}\left(Q_{m-1b}-Q_{mb}\right)$$

$$= e^{+i\frac{N}{\Delta t}\frac{(\Delta Q)M^{-1}(\Delta Q)}{2}} \int \frac{dP}{2\pi}\, e^{-i\frac{\Delta t}{N}\left(P - M^{-1}\frac{N}{\Delta t}\Delta Q\right)M\left(P - M^{-1}\frac{N}{\Delta t}\Delta Q\right)}$$

shift integral variable by

<u>Integral</u> gives some value <u>independent</u> of what Q_m, Q_{m-1} are.

$$= \tilde{N}\, e^{\frac{i\Delta t}{N}\left(\frac{\cancel{\Delta t}\,N(Q_m-Q_{m-1})_a}{\cancel{N}\,\Delta t}\right)\frac{M^{-1}_{ab}}{2}\left(\frac{N(Q_m-Q_{m-1})_b}{\Delta t}\right)}$$

note that $\dot{Q} \simeq \frac{\Delta Q}{\Delta t} = \frac{(Q_m - Q_{m-1})}{\Delta t/N} =$

so this is $\tilde{N}\, e^{\frac{i\Delta t}{N}\dot{Q}_a \frac{M^{-1}_{ab}}{2}\dot{Q}_b}$

Total: $\langle Q_m | U(\Delta t/N) | Q_{m-1}\rangle = \tilde{N}\, e^{\frac{i\Delta t}{N}\left(\frac{\dot{Q}M^{-1}\dot{Q}}{2} - V(Q)\right)}$

Look! That's the <u>Lagrangian</u>!!

$$Z(J) = \int \mathcal{D}Q\, \langle 0|Q_N\rangle\langle Q_0|0\rangle\, \tilde{N}^{\#} \exp i\int_{t_i}^{t_f} dt'\, L(Q(t'),\dot{Q}(t'))$$

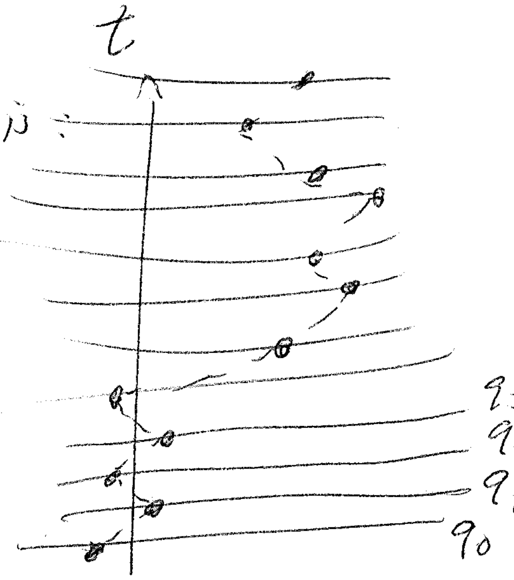
For field theory, $L = \int d^3x\, \mathcal{L}$, so it's $\exp i\int_{t_i}^{t_f} d^4x\, \mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$

$L_q P$

Path Int: Interpretation

For Quantum Mechanics of 1 variable:

$\int dq_0\, dq_1\, dq_2 \dots dq_N$ is what $\int \mathcal{D}q$ means.



<u>one</u> entry in integral (one choice of $\int dq_0 \dots dq_N$) is:

one choice of q at each time.

<u>Because of</u> $e^{\frac{i\Delta t}{N}\dot{Q}^2}$ term,

need $\dot{Q} \lesssim \sqrt{\Delta N/\Delta t}$

$\Delta Q \sim \sqrt{\Delta t/N}$

Doesn't jump <u>arbitrarily</u>: size of jumps in time t is $\sim \sqrt{t}$

<u>Brownian</u> Path

Interpretation of $q_1 \dots q_N$ is "a possible time history"

or "a possible <u>path</u> my particle can take"

$\int \mathcal{D}Q$ = integral over all possible <u>paths</u> a particle can take.

Feynman introduced this in 1960's. Technically, limit as # of $\int$'s goes to ∞. <u>Mathematics</u>? — Mathematicians initially skeptical

"Functional Integral" — Eventually proved that such things are well defined mathematically.

For Finite Volume & Discrete-Space, also valid for QFT (Just QM with lots of generalized coord)

But <u>limit</u> (time-spacing $\to 0$ <u>and</u> space-spacing $\to 0$) <u>not</u> clearly well posed.

Path $\int$ in QFT

$\int \mathcal{D}Q$ for $Q = \{ \varphi(x,t) \}$ is

Integral over all possible classical field spacetime histories, that is, all choices of $\varphi(x,t)$ function.

Including those which don't obey classical field EQ.

$a \to 0$ limit: fields which are arbitrarily "bouncy" or "violent" on short distances as we saw: $\langle \varphi^2(0) \rangle \sim a^{-2}$.

Option 1: Try to Do the integrals.

Lattice: make spacetime discrete & finite

go to Euclidean time (we'll come to, maybe)

Do the 10^8 integrals by some numerical means.

Hard. But not as crazy as it sounds.

Option 2: try to find some systematic approximation method. We will look at Perturbation Theory.