Theoretische Physik I: Klassische Mechanik - Präsenzübung

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Aufgabe 8.1: Manual computing

Consider the following pair of first-order ODEs for the two functions $s(t)$ and $c(t)$:

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} = c(t) \\[2mm] \dfrac{\mathrm{d}c(t)}{\mathrm{d}t} = -s(t) \end{cases} \tag{8.1.1}$$

The initial values are $c(0) = 1$ and $s(0) = 0$.

8.1a)

Of course we know the solution: $c(t) = \cos(t)$ and $s(t) = \sin(t)$. First show that these are the exact solutions.

8.1b)

Now let us try out the approximation methods which we discussed in class. Get as far as you can, using whatever methods you can – paper, pocket calculator, Mathematica, etc.
First, try to use the Euler method. Recall that, for a set of variables $g_1, \ldots, g_n$ and differential equations

$$\frac{\mathrm{d}g_i}{\mathrm{d}t} = f_i(g_j) \tag{8.1.2}$$

the Euler update is:

$$g_i(t + \Delta t) = g_i(t) + (\Delta t)f_i(g_j(t)) \tag{8.1.3}$$

Using the spacing $\Delta t = 1$, evaluate $s(1), c(1)$.
Then find $s(2), c(2)$, $s(3), c(3)$, and $s(4), c(4)$.
How are the solutions behaving? You should be able to do this with pencil and paper.
Get out your calculator/computer and try again with $\Delta t = 0.2$; try to get to $t = 1$, and see how close the values of s, c are from the exact results.

8.1c)

Next, we try the second-order Runge-Kutta technique. For this technique, we solve the system of differential equations as follows:

$$g_{i,\mathrm{proj}}(t + \Delta t) = g_i(t) + (\Delta t)f_i(g_j(t))$$
$$g_i(t + \Delta t) = g_i(t) + \frac{\Delta t}{2}\left(f_i(g_j(t)) + f_i(g_{j,\mathrm{proj}}(t + \Delta t))\right) \tag{8.1.4}$$

Try this approach, again by hand, with $\Delta t = 1$, advancing the fields a few steps. Then try with $\Delta t = 0.2$, now using a computer, and try to get to $t = 1$.

8.1d)

Finally, we try the fourth-order Runge-Kutta approach, based on four estimates for the slope and three projected values of the functions:

$$f_{i,1} = f_i(g_j(t))$$
$$g_{i,1} = g_i(t) + (\Delta t/2)f_{i,1}$$
$$f_{i,2} = f_i(g_{j,1})$$
$$g_{i,2} = g_i(t) + (\Delta t/2)f_{i,2}$$
$$f_{i,3} = f_i(g_{j,2}) \tag{8.1.5}$$
$$g_{i,3} = g_i(t) + (\Delta t)f_{i,3}$$
$$f_{i,4} = f_i(g_{j,3})$$
$$g_i(t + \Delta t) = g_i(t) + \frac{\Delta t}{6}(f_{i,1} + 2f_{i,2} + 2f_{i,3} + f_{i,4})$$

Try it for $\Delta t = 1$ and see how well or badly it works! Then try it for $\Delta t = 0.2$ and see how well it works after 5 steps.