Theoretische Physik I: Klassische Mechanik - Präsenzübung

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Übungsblatt 9

Aufgabe 9.1: Oscillations

You can use any coordinates you wish, but if you use the wrong ones, you will suffer. Let's see what happens when we make a somewhat foolish choice of coordinates for the Simple Harmonic Oscillator, as an example of Legendre transforms, and of how coordinate transformations cause the momenta, and hence phase space, to transform.

Consider the SHO in two dimensions (x, y), with Lagrange function

$$L(x, y, \dot{x}, \dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{D}{2}\left(x^2 + y^2\right). \tag{9.1.1}$$

9.1a)

Find the canonical momenta p_x, p_y. Perform the Legendre transform to find $H(x, y, p_x, p_y)$. This should be easy.

9.1b)

Consider a new set of coordinates (r, s) defined as follows:

$$\begin{cases} r = x \\ s = x + y \end{cases} \tag{9.1.2}$$

Write out this relation as a matrix, that is,

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{9.1.3}$$

Express each element of the matrix as a derivative of a new coordinate with respect to an old coordinate, e.g., $m_{11} = \partial r / \partial x$.

9.1c)

Find expressions for (x, y) as functions of (r, s). You can either do this algebraically, or you can find the inverse of the matrix which you just found.

9.1d)

Express the Lagrangian in terms of r, s. What strange property do you notice – is there an $\dot{r}\dot{s}$ term?

9.1e)

Find the Hamiltonian by applying a Legendre transform to this Lagrangian. Is there a $p_r p_s$ term, and does it have the same sign as the $\dot{r}\dot{s}$ term in the Lagrangian?

9.1f)

Let's start instead with the Hamiltonian in terms of x, y, $H(x, y, p_x, p_y)$ which we found before. Use the fact that $\partial p_r/\partial p_x = \partial x/\partial r$ and similarly for the other components, use the form of the transformation from $(r, s) \to (x, y)$ to determine the transformation from $(p_x, p_y) \to (p_r, p_s)$. Apply this transformation, and the transformation on the coordinates, to transform $H(x, y, p_x, p_y) \to H(r, s, p_r, p_s)$. Check that you get the same result as you found by Legendre transforming $L(r, s, \dot{r}, \dot{s})$.

9.1g)

Find the Hamilton's equations for $\dot{r}, \dot{s}, \dot{p}_r, \dot{p}_s$. Does it *appear* that the evolution mixes r with s, that is, do the dynamics appear to be more complicated in these coordinates than they are in the original (x, y) coordinates?

(There is no need to solve the Hamilton's equations, just find their form.)

9.1h)

If you have plenty of time left over, try to find $\ddot{r}$ as a function of (r, s).

Much of what we will do with oscillations is to start with systems which have this kind of non-diagonal kinetic term, and to rework them so that they are simple. That is, we will start with systems which look like the (r, s)-coordinate version, and we will try to find the (x, y)-coordinate version of the problem.