Theoretische Physik I: Klassische Mechanik - Übungsblatt

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Übungsblatt 3

Deadline: 6.05. 23 Uhr online

Aufgabe 3.1: A straight path. 5p.

In the absence of a potential, a particle (of mass M) flies in a straight line at constant velocity v_0 (by Newton's first law). The particle starts at $x(0) = 0$ and evolves into $x(t_f) = x_f \equiv v_0 t_f$. Let's see that this solution strictly minimizes the action.

3.1a)

Write an expression for the action in 1 dimension.

3.1b)

Write $x(t) = v_0 t + f(t)$ and rewrite your expression for the action using this form. What are the boundary conditions for $f(t)$? Is this a particular case? (i.e. have we lost any generality by considering this form of $x(t)$?)

3.1c)

Show that the term linear in $\dot{f}(t)$ vanishes on integration, while the quadratic term, $\dot{f}^2(t)$ is nonnegative and is only zero if $f = 0$ everywhere.

Aufgabe 3.2: Let there be light. 7p.

3.2a)

Show that the shortest distance between two points is a straight line. To do this, work in 2 dimensions, and pick coordinates where the starting and ending points are $(0,0)$ and $(x_f, 0)$ and consider $y = y(x)$. Write an expression for the total distance in terms of $y(x)$, and show that it is minimized when $y(x) = 0$.

3.2b)

The time it takes light to travel a distance ℓ is ℓ/c in vacuum, but $n\ell/c$ in a medium with index of refraction n (and speed of light c/n).

Fermat's principle states that light going from point $\vec{r}_1$ to point $\vec{r}_2$ will take the path which takes the shortest time. Show that, in empty space, light travels in a straight line.

3.2c)

Next, consider light which starts at $(0, -y_0)$ and goes to $(x_f, +y_f)$ with y_0 and y_f positive distances. For $y < 0$ the index of refraction is $n = 1$, while for $y > 0$ it is $n = n_0 > 1$ (for glass, $n = 1.5$). Assume that the light crosses $y = 0$ at a point $(x_0, 0)$. Argue that the light takes a straight path from $(0, -y_0)$ to $(x_0, 0)$ and a straight path from $(x_0, 0)$ to (x_f, y_f). Find the total time the light takes, and use Fermat's principle to solve for x_0. Show that the incoming and outgoing angles obey Snell's law.

Aufgabe 3.3: It is never luck. 7p.

First, show that the perimeter of a circle with area A is smaller than the perimeter of a square with area A. Is this luck, or something deep?

Consider a general curve which we will write in polar coordinates as $r = r(\theta)$. Write an expression for the perimeter of the curve P and an expression for its area A. Applying a Lagrange multiplier for A, so that our "action" is $S = P - \lambda A$, find an equation for $r(\theta)$ which shows that it minimizes P at fixed A. Show that the circle $r = R$ is a solution, while anything with a corner is not.

Extra credit: Is $r = R$ the only solution? If not, what else solves the equation?