Theoretische Physik I: Klassische Mechanik - Übungsblatt

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Übungsblatt 9

Deadline: 17.06. 23 Uhr online

Aufgabe 9.1: Some brackets. 10p.

9.1a) 10p

Compute each of the following Poisson brackets. Here $\vec{L} = \vec{r} \times \vec{p}$ is the angular momentum. You may find them easiest to compute by using index notation and writing $L_i = \epsilon_{ijk} r_j p_k$. It should be possible to perform your computations using only the properties of the Poisson bracket, without performing any actual derivatives.

$$\Big[L_i, r_j\Big] \qquad \Big[L_i, p_j\Big] \qquad \Big[L_i, L_j\Big] \qquad \Big[L_i, L^2\Big] \tag{9.1.1}$$

Background: In general, the Poisson bracket of angular momentum L_i with any vector $\vec{v}$ is $[L_i, v_j] = \epsilon_{ijk} v_k$. Whereas, for any scalar quantity s, $[L_i, s] = 0$. You can see that the Poisson bracket with L_i has something to do with the rotation of the coordinates, in the same way that the Poisson bracket with H has something to do with time evolution. These issues will be revisited in your quantum mechanics class!

Aufgabe 9.2: A test on Liouville's theorem

Here we see how Liouville's theorem is compatible with the exponential growth of differences between similar initial conditions.

Return to the Lagrangian which we saw in the numerical exercise,

$$L(x, y, \dot{x}, \dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - D_1(x^4 + y^4) - D_2 x^2 y^2 \tag{9.2.1}$$

Here we will consider the problem numerically with the choice $m = 1$, $D_1 = 1$ and $D_2 = -1$.

9.2a) 2p

Find the Hamiltonian $H(x, y, p_x, p_y)$.

9.2b) 2p

Consider the following 5 initial values at $t = 0$:

$$\begin{aligned}
(x, y, p_x, p_y)_A &= (1, 0, 0, 1) \\
(x, y, p_x, p_y)_B &= (1.001, 0, 0, 1) \\
(x, y, p_x, p_y)_C &= (1, 0.001, 0, 1) \\
(x, y, p_x, p_y)_D &= (1, 0, 0.001, 1) \\
(x, y, p_x, p_y)_E &= (1, 0, 0, 1.001)
\end{aligned} \tag{9.2.2}$$

Calculate the values of (x, y, p_x, p_y) at the times: $t = 1$, $t = 3$, $t = 10$ numerically.

9.2c) 2p

Calculate each distance $||(x, p)_B - (x, p)_A||$, $||(x, p)_C - (x, p)_A||$ etc between A and each other point, at the initial time and at each successive time $t = 1, 3, 10$. Here the distance is $\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (p_{xB} - p_{xA})^2 + (p_{yB} - p_{yA})^2}$ etc. Do the distances increase with time?

9.2d) 2p

We can view the point $(x_A, y_A, p_{xA}, p_{yA})$ as the bottom corner of a parallelotope (an N-dimensional generalization of a parallelepiped) in 4-dimensional phase space (x, y, p_x, p_y), with (B, C, D, E) the four neighboring corners. The 4-volume of such a parallelotope is

$$V = \left| \mathrm{Det} \begin{bmatrix} (x_B - x_A) & (y_B - y_A) & (p_{xB} - p_{xA}) & (p_{yB} - p_{yA}) \\ (x_C - x_A) & (y_C - y_A) & (p_{xC} - p_{xA}) & (p_{yC} - p_{yA}) \\ (x_D - x_A) & (y_D - y_A) & (p_{xD} - p_{xA}) & (p_{yD} - p_{yA}) \\ (x_E - x_A) & (y_E - y_A) & (p_{xE} - p_{xA}) & (p_{yE} - p_{yA}) \end{bmatrix} \right| \tag{9.2.3}$$

Calculate this volume at each time from your numerically determined coordinates. Does the volume increase with time? How does its increase compare with the increase in the distance between points?

9.2e) 2p

If we made the initial distances small enough, and the numerical precision good enough, then the volume should not increase. Why not? How can we be sure?