Quantum Field Theory II
Homework 1

Due 3 November 2023

The purpose of this homework is to familiarize you with some necessary but also fun mathematical things we need for integrations.

1 Gamma function

The gamma function is defined, for positive n, as

$$\Gamma(n) \equiv \int_0^\infty x^{n-1} e^{-x}\, dx$$

(Think of n as the sum of the number of x's and dx's appearing in the integral.)

First, argue that $\Gamma(n)$ is well defined in terms of this integral expression for all n in the complex plane which satisfy $\mathrm{Re}(n) > 0$.

Second, use integration by parts to prove the recursion relation

$$\Gamma(n+1) = n\Gamma(n) \qquad \text{or} \qquad \Gamma(n) = \frac{\Gamma(n+1)}{n} \tag{1}$$

so long as $\mathrm{Re}(n) > 0$ (including for complex n).

We will *define* the function for n with smaller real part recursively through this recursion relation, so, eg,

$$\Gamma(i-3) = \frac{\Gamma(i-2)}{i-3} = \frac{\Gamma(i-1)}{(i-2)(i-3)} = \ldots = \frac{\Gamma(i+1)}{i(i-1)(i-2)(i-3)}\,,$$

and $\Gamma(i+1)$ is well defined by the integral expression.

Argue that this recursive definition works for any n except for the nonpositive integers. The gamma function is not defined at those points.

Show that $\Gamma(z)$ has a simple pole about $z = 0$ with residue 1. Hint: $\Gamma(z) = \Gamma(z+1)/z$. Show furthermore that $\Gamma(z)$ has a simple pole at every negative integer, with residue

$$\mathrm{Res}(\Gamma(-n)) = \frac{(-1)^n}{n!}\,.$$

2 Digamma function

Define

$$\psi(z) = \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz} = \frac{d \ln \Gamma(z)}{dz}\,. \tag{2}$$

Prove that, for n with positive real part,

$$\psi(n) = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} \ln(x) e^{-x} dx \,. \tag{3}$$

Next use integration by parts to establish the relationship:

$$\psi(n+1) = \frac{1}{n} + \psi(n) \,. \tag{4}$$

Next show that, for large positive real n,

$$\psi(n) \simeq \ln(n) \tag{5}$$

where $\simeq$ means that they are equal up to terms which go to zero in the large-n limit, eg, $\lim_{n\to\infty}(\psi(n) - \ln(n)) = 0$. (If you can't prove it, at least explain why it's true. Hint: for what x value is $x^{n-1}e^{-x}$ maximum, and how wide is that peak?)

Use the recursion relation and this limit to show that:

$$\psi(1) = \lim_{N\to\infty} \left(\ln(N) - \sum_{n=1}^{N} \frac{1}{n} \right) \equiv -\gamma_E \tag{6}$$

where the quantity in parentesis defines minus the Euler-Mascheroni constant.

Use this result and Taylor expansion to show that

$$\Gamma(1+\epsilon) \simeq 1 - \gamma_E \epsilon + \mathcal{O}(\epsilon^2) \qquad \text{and} \qquad \Gamma(\epsilon) \simeq \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \,. \tag{7}$$

3 Beta function

The function mathematicians call the Beta Function is:

$$B(\alpha,\beta) \equiv \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \,. \tag{8}$$

Show that this is well defined for $\alpha > 0$ and $\beta > 0$ and that it is symmetric, $B(\alpha,\beta) = B(\beta,\alpha)$.

You can look up that there is a closed-form relation for this function:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \,. \tag{9}$$

Rather than derive this form, we will instead see how it helps us when we do dimensional regularization.

Consider the integral

$$2\int_0^\infty \frac{q^{D-1+2A}dq}{(q^2+M^2)^B}$$

where D is the dimension of space, so the radial integration is $q^{D-1}dq$. Here we consider the possibility that A extra powers of q^2 occur in the numerator, and B powers of (q^2+M^2) are in the denominator. Assume $B > A + D/2$ so that the integral converges.

First introduce $x = q^2/M^2$. Rewrite the integral as

$$M^{\#}\int_0^\infty \frac{x^{A+(D/2)-1}dx}{(x+1)^B}\,.$$

What is the power # on M?

Now introduce $y = x/(1+x)$. Show that a change of variables to y as integration variable leads to an integral which is precisely in the form of a beta function. Evaluate it. Show that the gamma functions are all in the "principal domain" (positive arguments) precisely when $B > A + D/2$. Outside of this case, the function is only defined by analytic continuation, just like the gamma function.

The beauty of dimensional regularization is this possibility of defining integrals via analytical continuation, even where the integral expressions do not, properly speaking, converge.

4 Really doing a Dimensional Regularization Integral

Let's actually do an actually hard integration with Dimensional Regularization. This will force us to learn all of the associated tricks. The problem will present enough help along the way that you should be able to finish it successfully.

Suppose you had to perform the following integral:

$$I = \int \frac{d^Dq}{(2\pi)^D}\frac{(q\cdot p)^2}{(q^2+m^2)^3}\,. \tag{10}$$

The integral is expressed in Euclidean space, so Wick rotation has already been carried out.

4.1 Is it divergent?

Show that the integral diverges for $D \geq 4$, but converges for $D < 4$. We will therefore take $D = 4 - 2\epsilon$ and will treat ϵ as a positive infinitesimal value.

4.2 Dealing with the numerator

The numerator involves a dot product. Write this out explicitly as $(p \cdot q)^2 = p_\mu p_\nu q^\mu q^\nu$. Argue that the $p_\mu p_\nu$ factor can be moved outside the integral, so we need

$$I = p_\mu p_\nu I^{\mu\nu} \,, \qquad I^{\mu\nu} \equiv \int \frac{d^D q}{(2\pi)^D} \frac{q^\mu q^\nu}{(q^2+m^2)^3} . \tag{11}$$

Argue that $I^{\mu\nu}$ is a rank-2 tensor which does not depend on any 4-vector. (Why not?)

By Lorentz invariance (or Euclidean rotation invariance), the result for $I^{\mu\nu}$ must be rotation-invariant. What rotation-invariant rank-2 tensors, with no dependence on any 4-momentum, exist?

Use this argument to write (since we are Euclidean, $\eta^{\mu\nu} \to \delta^{\mu\nu}$)

$$I^{\mu\nu} = \delta^{\mu\nu} J \tag{12}$$

with J some unknown *scalar* value.

Consider the contraction $\delta_{\mu\nu} I^{\mu\nu}$. Show that

$$\delta_{\mu\nu} I^{\mu\nu} = DJ = \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{(q^2+m^2)^3} \,. \tag{13}$$

4.3 Angular integral

In general one can write

$$\int d^D q \; F(q^2) = A(D) \int_0^\infty F(q^2)\, q^{D-1} \; dq \,. \tag{14}$$

Here $A(D)$ is the area of the sphere in D dimensions. But what is that, for general D?

To find $A(D)$, consider the special case that $F(q^2) = \exp(-q^2/\sigma^2)$. Insert this choice into Eq. (14) and use the trick that $\int d^D q e^{-q^2/X} = (\int dq e^{-q^2/X})^D$, the same trick Gauß used, with $D = 2$ and $X = 2$, to solve the original Gaussian integral. Using

$$\int e^{-q^2/\sigma^2} dq = \sigma\sqrt{\pi} \tag{15}$$

$$\int_0^\infty e^{-q^2/\sigma^2} q^{D-1} \; dq = \frac{\sigma^D}{2} \int_0^\infty e^{-z} z^{\frac{D-2}{2}} \, dz = \frac{\sigma^D \Gamma(D/2)}{2} \tag{16}$$

to show that

$$A(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \,. \tag{17}$$

Recalling that $\Gamma(1/2) = \sqrt{\pi}$ and using the recursion relation from the first problem, show that this reproduces the known results for $D = 1, 2, 3, 4$.

4.4 Doing our integral

Use the expressions from the last section to write our desired integral as

$$
\begin{aligned}
I &= \frac{p^2}{D} \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{(q^2+m^2)^3} &(18)\\
&= \frac{p^2}{D} \frac{2}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty \frac{q^{D+1}}{(q^2+m^2)^3} dq &(19)
\end{aligned}
$$

and use the results from the last problem ($A = 1$ and $B = 3$) to write this in a final form.

Ideally you should also series expand this final form in small ϵ, finding the $1/\epsilon$ and constant terms. But this will be extra credit.