Quantum Field Theory II
Homework 2

Due 24 November 2023

All problems in this homework will refer to Yukawa theory with fields ϕ, ψ a real scalar and a Dirac spinor respectively, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_1^2}{2}\phi^2 + \bar{\psi}(i\not{\partial} - m_2)\psi - \frac{\lambda}{24}\phi^4 - y\phi\bar{\psi}\psi\,. \tag{1}$$

I will refer to λ as the scalar self-coupling and y as the Yukawa coupling.

1 Bare and renormalized

The bare Lagrangian is the one written above, with all fields and couplings labeled with a 0, eg, $\phi_0, \psi_0, y_0, \lambda_0, m_{1,0}, m_{2,0}$. The renormalized Lagrangian is a rewritten form of the bare Lagrangian:

$$\begin{aligned}\mathcal{L} \;=\;& \frac{1}{2}Z_\phi\partial_\mu\phi_r\partial^\mu\phi_r - \frac{m_{1,r}^2 + \delta m_1^2}{2}Z_\phi\phi_r^2 - \frac{\lambda_r + \delta\lambda}{24}Z_\phi^2\phi_r^4 \\ & + Z_\psi\bar{\psi}_r(i\not{\partial} - (m_{2,r} + \delta m_2))\psi_r - (y_r + \delta y)Z_\phi^{1/2}Z_\psi\phi_r\bar{\psi}_r\psi_r\,. \end{aligned} \tag{2}$$

Write an expression for the relation between each renormalized and each bare quantity, eg, $\phi_r = Z_\phi^{-1/2}\phi_0$ etc.

2 Divergent diagrams

In this problem we will only attempt to find which diagrams are divergent and what the factors of $1/\epsilon$ are in their 1-loop expressions.

Based on the power counting arguments we presented, find all diagrams which are divergent at 1-loop order. How divergent do you expect each diagram to be (eg, logarithmic, linear, quadratic)?

3 Self-energies

Write an expression for each scalar-field self-energy diagram (there should be two diagrams). Evaluate each in the $\overline{\text{MS}}$ scheme, but do not try to perform all integrals exactly – only try to determine the coefficient multiplying $1/\epsilon$ (which will be simpler). Labeling the incoming momentum as p, your answer should be of form

$$\Pi = (Ap^2 + B)\frac{1}{\epsilon}$$

where A, B are combinations of couplings and 4π type factors.

Repeat for the fermionic self-energy. Here you should find only one relevant diagram, and an answer of form

$$\Sigma = (A\not{p} + B)\frac{1}{\epsilon}\,.$$

You should find that $B \propto m_2$.

4 Vertex

Draw all one-loop amputated 1PI vertex correction diagrams – separately for the ϕ^4 vertex and the $\phi\bar{\psi}\psi$ diagram. You should find two diagrams for the ϕ^4 case but only one diagram for the Yukawa case.

Compute these diagrams in dimensional regularization. Again, only try to determine the $1/\epsilon$ coefficient, not any finite or momentum-dependent corrections beyond this level.

In the *next* homework, we will see what these diagrams imply for the anomalous dimensions and beta functions of the theory.

5 Fermion mass

Something curious happens with the momentum-independent term in the fermionic self-energy. It could have been divergent already for $D = 3$, but you should find that this divergence cancels on angular integration, and the first pole as a function of D occurs for $D = 4$, indicating that the diagram is truly log divergent. And the log divergence is proportional to the mass m_2, meaning that, if $m_2 = 0$, there is no such contribution. In other words, if the bare mass $m_2 = 0$, then the renormalized mass is also zero. (This was *not* true for the scalar field, where the diagram diverges already for $D = 2$, and the log divergence in $D = 4$ depends both on m_1^2 and m_2^2.)

This is actually a consequence of a symmetry which appears when we set $m_2 = 0$. Consider the following (discrete) transformation:

$$\begin{aligned} \psi &\rightarrow \gamma^5 \psi \\ \phi &\rightarrow -\phi \end{aligned}$$

First, show that $\bar{\psi} \rightarrow -\bar{\psi}\gamma^5$. (Why is there a $-$ sign here?)
Next, show that $\bar{\psi}\not{\partial}\psi$ is invariant under this symmetry, and so is $\phi\bar{\psi}\psi$, but $\bar{\psi}\psi$ is not. Therefore, if this symmetry is present, it is impossible for a mass term to appear in the Lagrangian. Since the symmetry is present in the renormalized as well as the bare theory, the fermion must be *exactly* massless at the renormalized, as well as bare, level.

This is a discrete case of a *chiral symmetry*. Such symmetries will be important in QCD.