Quantum Field Theory II
Homework 2 Solutions

All problems in this homework will refer to Yukawa theory with fields ϕ, ψ a real scalar and a Dirac spinor respectively, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_1^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m_2)\psi - \frac{\lambda}{24}\phi^4 - y\phi\bar{\psi}\psi\,. \tag{1}$$

I will refer to λ as the scalar self-coupling and y as the Yukawa coupling.

1 Bare and renormalized

The bare Lagrangian is the one written above, with all fields and couplings labeled with a 0, eg, $\phi_0, \psi_0, y_0, \lambda_0, m_{1,0}, m_{2,0}$. The renormalized Lagrangian is a rewritten form of the bare Lagrangian:

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}Z_\phi\partial_\mu\phi_r\partial^\mu\phi_r - \frac{m_{1,r}^2 + \delta m_1^2}{2}Z_\phi\phi_r^2 - \frac{\lambda_r + \delta\lambda}{24}Z_\phi^2\phi_r^4 \\ & + Z_\psi\bar{\psi}_r(i\partial\!\!\!/ - (m_{2,r} + \delta m_2))\psi_r - (y_r + \delta y)Z_\phi^{1/2}Z_\psi\phi_r\bar{\psi}_r\psi_r\,. \end{aligned} \tag{2}$$

Write an expression for the relation between each renormalized and each bare quantity, eg, $\phi_r = Z_\phi^{-1/2}\phi_0$ etc.

1.1 Solution

$$\phi_r = Z_\phi^{-1/2}\phi_0 \tag{3}$$
$$\psi_r = Z_\psi^{-1/2}\psi_0 \tag{4}$$
$$m_{1r}^2 = m_{10}^2 - \delta m_1^2 \tag{5}$$
$$m_{2r} = m_{20} - \delta m_2 \tag{6}$$
$$\lambda_r = \lambda_0 - \delta\lambda \tag{7}$$
$$y_r = y_0 - \delta y\,. \tag{8}$$

2 Divergent diagrams

In this problem we will only attempt to find which diagrams are divergent and what the factors of $1/\epsilon$ are in their 1-loop expressions.

Based on the power counting arguments we presented, find all diagrams which are divergent at 1-loop order. How divergent do you expect each diagram to be (eg, logarithmic, linear, quadratic)?

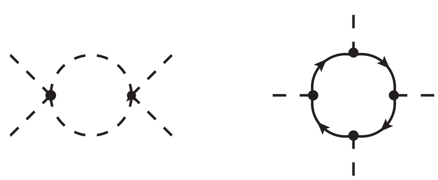
2.1 Solution

A diagram is divergent if the sum of 1 for each external scalar line plus 3/2 for each external fermionic line is at most 4. Since the number of fermionic lines must be even by angular momentum conservation and since the one and three scalar cases are zero by the discrete $\phi \to -\phi$ symmetry, we have:

2 scalars

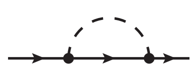
$D = 4 - 2 = 2$ quadraticdivergent

4 scalars



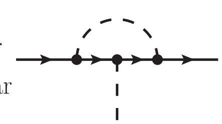
$D = 4 - 4 = 0$ log − divergent

2 spinors



$D = 4 - 2 \times \frac{3}{2} = 1$ linear − divergent

2 − spinor
1 − scalar



$D = 4 - 2 * (3/2) - 1 = 0$ Log − divergent

3 Self-energies

Write an expression for each scalar-field self-energy diagram (there should be two diagrams). Evaluate each in the $\overline{\text{MS}}$ scheme, but do not try to perform all integrals exactly – only try to determine the coefficient multiplying $1/\epsilon$ (which will be simpler). Labeling the incoming momentum as p, your answer should be of form

$$\Pi = (Ap^2 + B)\frac{1}{\epsilon}$$

where A, B are combinations of couplings and 4π type factors.

Repeat for the fermionic self-energy. Here you should find only one relevant diagram, and an answer of form

$$\Sigma = (A\not{p} + B)\frac{1}{\epsilon}\,.$$

You should find that $B \propto m_2$.

3.1 Solution

Since the propagator is $i/(p^2 - m^2)$ and

$$\frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}\,(-i\Pi)\,\frac{i}{p^2 - m^2} + \ldots \Longrightarrow \frac{i}{p^2 - m^2 - \Pi} \tag{9}$$

we define Π to be i times the amputated diagram. Therefore the scalar contribution is

$$\begin{aligned} \Pi_{\text{bosonloop}}(p) &= (i)\frac{-i\lambda}{2}\int \frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2 - m_1^2 + i\epsilon} \\ &= \frac{\lambda}{2}\int \frac{d^D\ell_E}{(2\pi)^D}\frac{1}{\ell_E^2 + m_1^2} \end{aligned} \tag{10}$$

while the fermionic contribution is

$$\begin{aligned} \Pi_{\text{fermionloop}}(p) &= (i)(-iy)^2(-1)\int \frac{d^D\ell}{(2\pi)^D}\,\mathrm{Tr}\,\frac{i(\not{p} + \not{\ell} + m_2)i(\not{\ell} + m_2)}{((p+\ell)^2 - m_2^2 + i\epsilon)(\ell^2 - m_2^2 + i\epsilon)} \\ &= -i4y^2\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^2 + \ell\cdot p + m_2^2}{((p+\ell)^2 - m_2^2 + i\epsilon)(\ell^2 - m_2^2 + i\epsilon)} \\ &= -4iy^2\int_0^1 dx \int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^2 + (1-2x)\ell\cdot p - x(1-x)p^2 + m_2^2}{(\ell^2 - m_2^2 + x(1-x)p^2 + i\epsilon)^2} \end{aligned} \tag{11}$$

where I performed the trace and combined denominators with a Feynman parameter. We can drop the $(1-2x)\ell\cdot p$ term, which vanishes on angular integration and is odd under $x \to 1-x$. I will also rewrite the numerator as

$$\frac{\ell^2 - x(1-x)p^2 + m_2^2}{(\ell^2 - m_2^2 + x(1-x)p^2)^2} = \frac{1}{\ell^2 - m_2^2 + x(1-x)p^2} + 2\frac{m_2^2 - x(1-x)p^2}{(\ell^2 - m^2 + x(1-x)p^2)^2}\,. \tag{12}$$

We are now ready to Wick rotate, introducing a factor of i and also flipping the sign on ℓ^2:

$$\Pi_{\text{fermionloop}}(p) = 4y^2\int_0^1 dx \int \frac{d^D\ell_E}{(2\pi)^D}\left[\frac{2m^2 - 2x(1-x)p^2}{(\ell_E^2{+}m^2 - x(1{-}x)p^2)^2} - \frac{1}{\ell_E^2{+}m^2 - x(1{-}x)p^2}\right] \tag{13}$$

where the first term is log divergent and the second is quadratically divergent.

Next we use that

$$\begin{aligned}\int \frac{d^D\ell_E}{(2\pi)^D}\frac{1}{\ell^2+A} &= \frac{A^{\frac{D-2}{2}}\Gamma(1-D/2)}{(4\pi)^{D/2}\Gamma(1)} \\ &= \frac{A}{16\pi^2}\left[-\frac{1}{\epsilon} + O(1)\right] \qquad (14)\\ \int \frac{d^D\ell_E}{(2\pi)^D}\frac{1}{(\ell^2+A)^2} &= \frac{A^{\frac{D-4}{2}}\Gamma(2-D/2)}{(4\pi)^{D/2}\Gamma(2)} \\ &= \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} + O(1)\right). \qquad (15)\end{aligned}$$

Note that the first expression first diverges at $1 - D/2 = 0$ which is $D = 2$, which is how we identify it to be quadratically divergent. The coefficient on $1/\epsilon$ in the second expression does not depend on A, which is convenient.

Applying these expressions, we find that

$$\begin{aligned}\Pi_{\text{bosonloop}} &= -\frac{\lambda}{2}\frac{m_1^2}{16\pi^2}\frac{1}{\epsilon} \qquad (16)\\ \Pi_{\text{fermionloop}} &= \frac{4y^2}{16\pi^2}\frac{1}{\epsilon}\left[\int_0^1 dx\left(2m_2^2 + 2x(1-x)p^2 + m^2 - x(1-x)p^2\right)\right] \\ &= \frac{y^2}{16\pi^2}\frac{1}{\epsilon}\left(12m_2^2 - 2p^2\right)\,. \qquad (17)\end{aligned}$$

In total,

$$\Pi(p) = \frac{1}{16\pi^2}\frac{1}{\epsilon}\left[-2y^2\,p^2 + 12y^2m_2^2 - \frac{\lambda}{2}m_1^2\right]. \tag{18}$$

Now let's repeat the analysis for the fermion. We similarly find that

$$\begin{aligned}\Sigma(p) &= (i)(-iy)^2\int\frac{d^D\ell}{(2\pi)^D}\frac{i(\not\ell+\not p+m_2)i}{((\ell+p)^2-m_2^2+i\epsilon)(\ell^2-m_1^2+i\epsilon)}\\ &= iy^2\int_0^1 dx\int\frac{d^D\ell}{(2\pi)^D}\frac{\not\ell+x\not p-m_2}{(\ell^2-xm_1^2-(1-x)m_2^2-x(1-x)p^2+i\epsilon)^2}\end{aligned} \tag{19}$$

where we combined denominators using the Feynman trick. The term proportional to $\not\ell$ is odd in ℓ and therefore integrates to zero, which shows that the diagram is at most log divergent. Handling the denominator is the same as in the scalar case; we Wick rotate and apply Eq. (15), and we find

$$\begin{aligned}\Sigma(p) &= -y^2\int_0^1 dx\int\frac{d^D\ell_E}{(2\pi)^D}\frac{x\not p-m_2}{(\ell_E^2+xm_1^2+(1-x)m_2^2+x(1-x)p^2)^2}\\ &= -\frac{y^2}{16\pi^2}\left(\frac{1}{\epsilon}+O(1)\right)\left(\frac{1}{2}\not p-m_2\right).\end{aligned} \tag{20}$$

The constant term is proportional to m_2 and vanishes if $m_2 = 0$.

4 Vertex

Draw all one-loop amputated 1PI vertex correction diagrams – separately for the ϕ^4 vertex and the $\phi\bar\psi\psi$ diagram. You should find two diagrams for the ϕ^4 case but only one diagram for the Yukawa case.

Compute these diagrams in dimensional regularization. Again, only try to determine the $1/\epsilon$ coefficient, not any finite or momentum-dependent corrections beyond this level.

In the *next* homework, we will see what these diagrams imply for the anomalous dimensions and beta functions of the theory.

4.1 Solution

The all-scalar 4-point diagram gives

$$3\frac{(-i\lambda)^2}{2}\int\frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2-m_1^2+i\epsilon}\frac{i}{(\ell+p)^2-m_1^2+i\epsilon} \tag{21}$$

$$=\frac{3i\lambda^2}{2}\int_0^1 dx\int\frac{d^D\ell_E}{(2\pi)^D}\frac{1}{(\ell_E^2+m_1^2+x(1-x)p^2)^2} \tag{22}$$

$$\simeq\frac{3i\lambda^2}{2}\frac{1}{16\pi^2}\left(\frac{1}{\epsilon}+O(1)\right) \tag{23}$$

where p is the sum of two incoming momenta and the factor of 3 is a sum over which pair of incoming momenta it is. Because the $1/\epsilon$ part of Eq. (15) doesn't depend on A, the diagram doesn't depend on p which is why I have been sloppy about identifying it correctly.

Similarly, for the fermionic loop, we actually have to sum over the six orderings that the external lines can attach to the loop. Each gives a contribution of form

$$(-)(-iy)^4\int\frac{d^D\ell}{(2\pi)^D}\,\mathrm{Tr}\,\frac{i(\not\ell+\not p_1+m)}{(\ell+p_1)^2-m^2+i\epsilon}\,\frac{i(\not\ell+\not p_2+m)}{(\ell+p_2)^2-m^2+i\epsilon}\,\frac{i(\not\ell+\not p_3+m)}{(\ell+p_3)^2-m^2+i\epsilon}\,\frac{i(\not\ell+m)}{\ell^2-m^2+i\epsilon} \tag{24}$$

where $p_{1,2,3}$ are linear combinations of external momenta. If we only try to keep track of the terms with the most powers of ℓ in this, the trace becomes $4(\ell^2)^2/((\ell^2)+A)^4$ which has the same UV behavior as Eq. (15). Including the aforementioned factor of 6 from the orderings and the i from Wick rotation, we find

$$-24iy^4\frac{1}{16\pi^2}\left(\frac{1}{\epsilon}+O(1)\right) \tag{25}$$

where the $O(1)$ will be some complicated function of the invariants built from the external momenta which we will definitely not try to compute.

Note that the λ^2 term has the opposite sign of the $-i\lambda$ leading order contribution, while the y^4 term has the same sign. They will therefore contribute with opposite sign to the beta function, as we will see. The origin of this sign comes down to the $-$ sign which occurs in fermion loops due to the fermionic nature of the field, or equivalently the Grassmann nature of the fermionic integration.

Finally there is the Yukawa coupling. The diagram has as its Feynman rule

$$(-iy)^3 \int \frac{d^D\ell}{(2\pi)^D} \frac{i}{\ell^2 - m_1^2 + i\epsilon} \frac{i(\not{\ell} + \not{p} + m_2)}{(\ell+p)^2 - m_2^2 + i\epsilon} \frac{i(\not{\ell} + \not{p} + \not{q} + m_2)}{(\ell+p+q)^2 - m_2^2 + i\epsilon} \tag{26}$$

which again keeping the maximum power of ℓ is

$$\begin{aligned} &\simeq y^3 \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^2}{(\ell^2 - A)^3} \\ &= iy^3 \int \frac{d^D\ell_E}{(2\pi)^D} \frac{\ell_E^2}{(\ell_E^2 + A)^3} \\ &= iy^3 \frac{1}{16\pi^2} \left(\frac{1}{\epsilon} + O(1) \right). \end{aligned} \tag{27}$$

Again, to express A correctly, we would actually need to combine denominators with two Feynman parameters and we would find a complicated functional dependence on (p, q) the external momenta, but this all only appears in the $O(1)$ part and does not affect the divergent part of the diagram or the needed counterterm. The sign is opposite the leading-order $-iy$ contribution, just as for the λ^2 correction to λ.

5 Fermion mass

Something curious happens with the momentum-independent term in the fermionic self-energy. It could have been divergent already for $D = 3$, but you should find that this divergence cancels on angular integration, and the first pole as a function of D occurs for $D = 4$, indicating that the diagram is truly log divergent. And the log divergence is proportional to the mass m_2, meaning that, if $m_2 = 0$, there is no such contribution. In other words, if the bare mass $m_2 = 0$, then the renormalized mass is also zero. (This was *not* true for the scalar field, where the diagram diverges already for $D = 2$, and the log divergence in $D = 4$ depends both on m_1^2 and m_2^2.)

This is actually a consequence of a symmetry which appears when we set $m_2 = 0$. Consider the following (discrete) transformation:

$$\begin{aligned} \psi &\rightarrow \gamma^5 \psi \\ \phi &\rightarrow -\phi \end{aligned}$$

First, show that $\bar{\psi} \to -\bar{\psi}\gamma^5$. (Why is there a $-$ sign here?)
Next, show that $\bar{\psi}\partial\!\!\!/\psi$ is invariant under this symmetry, and so is $\phi\bar{\psi}\psi$, but $\bar{\psi}\psi$ is not. Therefore, if this symmetry is present, it is impossible for a mass term to appear in the Lagrangian. Since the symmetry is present in the renormalized as well as the bare theory, the fermion must be *exactly* massless at the renormalized, as well as bare, level.

This is a discrete case of a *chiral symmetry*. Such symmetries will be important in QCD.

5.1 Solution

Consider $\psi \to \gamma_5\psi$. Then

$$\psi^\dagger \to \psi^\dagger\gamma_5^\dagger = \psi^\dagger\gamma_5 \tag{28}$$

because γ_5 is Hermitian. (I freely interchange the notation $\gamma^5 = \gamma_5$ to keep the 5 away from the $\dagger$.) By definition

$$\overline{\psi} = \psi^\dagger\gamma^0 \to \psi^\dagger\gamma_5\gamma^0 = -\psi^\dagger\gamma^0\gamma^5 = -\overline{\psi}\gamma^5\,. \tag{29}$$

Therefore

$$\begin{aligned}\bar{\psi}\gamma^\mu\partial_\mu\psi &\to -\bar{\psi}\gamma^5\gamma^\mu\partial_\mu\gamma^5\psi \\ &= -\bar{\psi}\gamma^5\gamma^\mu\gamma^5\partial_\mu\psi \\ &= +\bar{\psi}\gamma^\mu\partial_\mu\psi \end{aligned} \tag{30}$$

and this term is unchanged. However,

$$\bar{\psi}\psi \to -\bar{\psi}\gamma^5\gamma_5\psi = -\bar{\psi}\psi \tag{31}$$

flips sign. If ϕ also flips sign then $\phi\bar{\psi}\psi \to \phi\bar{\psi}\psi$ without sign change. All other Lagrangian terms are ϕ-even so this transformation leaves the $m_2 = 0$ Lagrangian unchanged. However the transformation would take $m\bar{\psi}\psi \to -m\bar{\psi}\psi$ so a mass term is not compatible with this symmetry transformation.

If the Lagrangian respects the symmetry then every term in every Feynman diagram must respect the symmetry and the result of any diagram must also respect it, meaning that no mass can be generated. This ensures that any mass renormalization must be proportional to m_2 as the only Lagrangian term which does not respect the symmetry.