Quantum Field Theory II
Homework 3

15 December 2023

1 Renormalization of Yukawa Theory

In the previous homework we considered the Yukawa theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m_1^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m_2)\psi - \frac{\lambda}{24}\phi^4 - y\phi\bar{\psi}\psi\,. \tag{1}$$

Consider calculating this theory within the minimal-subtraction dimensional regularization scheme $\overline{\mathrm{MS}}$.

1.1 Warm-up

Consider the integral which appeared repeatedly in the prevous homework:

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2+A)^2}\,. \tag{2}$$

First argue that, on dimensional grounds, we must have forgotten a μ^{4-D} in front, so that the total dimension comes out the same as it would in $D = 4$ dimensions. Then argue that the result will be proportional to $A^{\frac{D-4}{2}}$. From these facts show that the analysis from last time,

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2+A)^2} \simeq \frac{1}{16\pi^2}\frac{1}{\epsilon} \tag{3}$$

should more properly have been

$$\mu^{4-D}\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2+A)^2} \simeq \frac{1}{16\pi^2}\frac{1}{\epsilon}\left(\frac{\mu^2}{A}\right)^{\frac{4-D}{2}}\,. \tag{4}$$

Using $A^\epsilon = e^{\epsilon\ln(A)} \simeq 1 + \epsilon\ln(A)$, show that this equals

$$\frac{1}{16\pi^2}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{A}\right) \tag{5}$$

up to $\mathcal{O}(\epsilon)$ corrections. That is, whenever one finds a $1/\epsilon$ factor, it must accompany a $\ln(\mu^2/A)$ where A is some combination of momenta or energy-invariants from the problem at hand. This will be enough for us to determine the μ dependence of the diagrams we examined last time.

1.2 Anomalous dimensions

Let us see how to compute anomalous dimensions.

Use the relation, found in the last homework, between ϕ and ϕ_0 to rewrite the bare propagator

$$G_0(p) = \int d^D x e^{ip_\mu x^\mu} \langle 0|\mathrm{T}(\phi_0(x)\phi_0(0))|0\rangle \tag{6}$$

as $Z_\phi G(p)$ with $G(p)$ the renormalized correlator. This is the correlator with the $1/\epsilon$ term removed.

The self-energy we computed in the last homework is easiest to use if we use the inverse propagator

$$G_0^{-1} = Z_\phi^{-1} G^{-1}(p) = Z_\phi^{-1}\left(p^2 - m^2 - \Pi(p)\right). \tag{7}$$

Use the expression we found for $\Pi(p)$ last time, and the fact that the bare correlator has no μ dependence, to evaluate the μ dependence of Z_ϕ and therefore the anomalous dimension. Use the same reasoning to find the anomalous dimension of the spinor.

1.3 Coupling

Use the Callan-Symanzik equation and the vertex corrections from the last homework, along with the anomalous dimensions from above, to determine the beta functions β_λ and β_y.

2 Renormalization Group and QED

We found in class that the beta function of QED is:

$$\frac{\mu\, de}{d\mu} = \frac{4}{3}\frac{e^3}{16\pi^2}. \tag{8}$$

Assume that this is true at all scales. (It's not – this is if there are only electrons!!) Use that, for $\mu = 511$KeV, that

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}\,. \tag{9}$$

Find an explicit expression for e or α as a function of μ and determine the value of μ for which e diverges. Compare this scale to the Planck scale, the scale by which gravity must become strongly coupled, which is 1.22×10^{19}GeV.

Actually, QED gets embedded into the Standard Model and becomes "hypercharge." Above the electroweak scale $\mu \sim 246$GeV, the "hyper"fine structure constant is $\alpha = 1/98$ and the beta function, featuring electrons, their heavier partners, quarks, and Higgs bosons, reads:

$$\frac{\mu\, de}{d\mu} = \frac{41}{6}\frac{e^3}{16\pi^2}\,. \tag{10}$$

(It's a long story to see where 41/6 comes from – let's not talk about it today.) For THIS expression, what is the scale where the coupling diverges? Is it still above the Planck scale?

3 Renormalization group and Banks-Zacks

Write $t = \ln(\mu^2)$, so $\mu^2 dx/d\mu^2 = dx/dt$. It's simpler and more compact to study renormalization group in terms of t, and we will work in the notation where μ^2, rather than μ, is used – this is common in the modern literature.

In QCD the beta function, expressed in terms of $a = \alpha/4\pi = g^2/16\pi^2$ rather than g, can be written:

$$\frac{d\,a}{dt} = -\beta_0 a^2 - \beta_1 a^3\,. \tag{11}$$

According to `https://arxiv.org/pdf/1701.01404.pdf`, the values of β_0 and β_1 for N_c-color, N_f-flavor QCD are:

$$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f\,, \tag{12}$$

$$\beta_1 = \frac{34}{3}N_c^2 - \frac{10}{3}N_c N_f - 4\frac{N_c^2-1}{4N_c}N_f. \tag{13}$$

(In comparison to the reference, I used $C_A = N_c$, $T_F = 1/2$ and $C_F = (N_c^2 - 1)/(2N_c)$. Here T_F is what we called $C(F)$ in class, and C_F is what we called $C_2(F)$. These are also common notation choices, I don't know why.)

For $N_c = 3$, for what values of N_f are $\beta_0 > 0$ but $\beta_1 < 0$? In this range, the beta function has a zero at finite a value a_0. What is the value of a_0? Are there any N_c values for which this zero occurs where a_0 is small?

See if you can compute the complete t dependence of a assuming that $a(t = 0)$ lies between 0 and a_0. If you cannot, then find the behavior of $a(t)$ just in the vicinity of a_0.

4 Group theory

4.1 Fundamental representation

In carrying out some calculation, you find yourself needing to perform two group-theory calculations, in SU(3) gauge theory:

$$\text{Answer } 1 = \operatorname{Tr} T^A T^A T^B T^B \,, \tag{14}$$

$$\text{Answer } 2 = \operatorname{Tr} T^A T^B T^A T^B \,. \tag{15}$$

Here $T^A = \frac{\lambda^A}{2}$ are the fundamental-representation generators of the SU(3) Lie algebra, which are half the Gell-Mann matrices. Sums over repeated indices are implicit as usual.

First, carry out each calculation using the group-theory tricks we learned, and evaluate them using:

$$C[F] = \frac{1}{2} \tag{16}$$

$$C[A] = N_c = 3 \tag{17}$$

$$d_F = N_c = 3 \tag{18}$$

$$d_A = N_c^2 - 1 = 8 \tag{19}$$

$$C_2[F] = \frac{d_A C[F]}{d_F} = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}. \tag{20}$$

Second, carry them out using the explicit expressions for each Gell-Mann matrix, by actually conducting all of the matrix multiplications, sums, and traces involved. If I were you I would do this using Mathematica, not by hand, but you are welcome to do it by hand if you really have to.

4.2 Six representation

Oh, but wait! The particles you THOUGHT were in the fundamental representation are actually in the symmetric tensor or 6 representation! This is the representation containing $|rr\rangle$ the state with two red quarks, and all states you arrive at through raising and lowering operators from this state, eg, $(|rg\rangle + |gr\rangle)/\sqrt{2}$, $|gg\rangle$, etc. We know that this rep has dimension $d_R = 6$ and Dynkin index (trace normalization) $C[R] = 5/2$, that is, $\mathrm{Tr}\, 1 = 6$ and $\mathrm{Tr}\, T^A T^B = (5/2)\delta_{AB}$. Can you carry out the same calculations as before, for this rep?

For extra credit, if you are really hard-line, see if you can find somehow the actual T^A matrices for this representation, and carry out the calculation directly. I recommend that you not attempt this extra credit, but if you want to you can try.