Quantum Field Theory II
Homework 3 Solutions

1 Renormalization of Yukawa Theory

In the previous homework we considered the Yukawa theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_1^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m_2)\psi - \frac{\lambda}{24}\phi^4 - y\phi\bar{\psi}\psi\,. \tag{1}$$

Consider calculating this theory within the minimal-subtraction dimensional regularization scheme $\overline{\mathrm{MS}}$.

1.1 Warm-up

Consider the integral which appeared repeatedly in the prevous homework:

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2+A)^2}\,. \tag{2}$$

First argue that, on dimensional grounds, we must have forgotten a μ^{4-D} in front, so that the total dimension comes out the same as it would in $D = 4$ dimensions. Then argue that the result will be proportional to $A^{\frac{D-4}{2}}$. From these facts show that the analysis from last time,

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2+A)^2} \simeq \frac{1}{16\pi^2}\frac{1}{\epsilon} \tag{3}$$

should more properly have been

$$\mu^{4-D}\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2+A)^2} \simeq \frac{1}{16\pi^2}\frac{1}{\epsilon}\left(\frac{\mu^2}{A}\right)^{\frac{4-D}{2}}\,. \tag{4}$$

Using $A^\epsilon = e^{\epsilon\ln(A)} \simeq 1 + \epsilon\ln(A)$, show that this equals

$$\frac{1}{16\pi^2}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{A}\right) \tag{5}$$

up to $\mathcal{O}(\epsilon)$ corrections. That is, whenever one finds a $1/\epsilon$ factor, it must accompany a $\ln(\mu^2/A)$ where A is some combination of momenta or energy-invariants from the problem at hand. This will be enough for us to determine the μ dependence of the diagrams we examined last time.

1.1.1 Solution

The general prescription for dimensional regularization is

$$\int \frac{d^4\ell}{(2\pi)^4} \to \mu^{4-D} \int \frac{d^D\ell}{(2\pi)^D} . \tag{6}$$

This is to ensure that the dimensionality of all integrals and couplings stays the same. Therefore the factors of μ are clear. In evaluating the integral

$$\mu^{4-D} \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 + A)^2} \tag{7}$$

the first step is to replace $\ell^2 \to \ell^2/A$ which is dimensionless, which by direct computation or on dimensional grounds will bring a factor of $A^{\frac{D-4}{2}}$ out front:

$$\left(\frac{\mu^2}{A}\right)^{\frac{4-D}{2}} \int \frac{d^D\tilde{\ell}}{(2\pi)^D(\tilde{\ell}^2 + 1)^2} \tag{8}$$

where the latter integrand is dimensionless and equals $(1/16\pi^2)(1/\epsilon + K)$ with K a constant which is typically absorbed by the $\overline{\epsilon}$ definition. This reproduced Eq. (4) above.

We then need to expand

$$\frac{1}{\epsilon}\left(\frac{\mu^2}{A}\right)^{\epsilon} = \frac{1}{\epsilon} e^{\epsilon \ln(\mu^2/A)} \simeq \frac{1}{\epsilon}\left(1 + \epsilon \ln \frac{\mu^2}{A}\right) = \frac{1}{\epsilon} + \ln \frac{\mu^2}{A} . \tag{9}$$

We therefore verify the general property that $1/\epsilon$ always accompanies a $\ln(\mu^2/A)$ with A some energy-squared associated with the physics of the problem.

1.2 Anomalous dimensions

Let us see how to compute anomalous dimensions.

Use the relation, found in the last homework, between ϕ and ϕ_0 to rewrite the bare propagator

$$G_0(p) = \int d^D x e^{ip_\mu x^\mu} \langle 0 | \mathrm{T}(\phi_0(x)\phi_0(0)) | 0 \rangle \tag{10}$$

as $Z_\phi G(p)$ with $G(p)$ the renormalized correlator. This is the correlator with the $1/\epsilon$ term removed.

The self-energy we computed in the last homework is easiest to use if we use the inverse propagator

$$G_0^{-1} = Z_\phi^{-1} G^{-1}(p) = Z_\phi^{-1} \left(p^2 - m^2 - \Pi(p)\right). \tag{11}$$

Use the expression we found for $\Pi(p)$ last time, and the fact that the bare correlator has no μ dependence, to evaluate the μ dependence of Z_ϕ and therefore the anomalous dimension. Use the same reasoning to find the anomalous dimension of the spinor.

1.2.1 Solution

We start with

$$G_0^{-1} = Z_\phi^{-1} \left[p^2 - m^2 - \Pi(p)\right] \tag{12}$$

$$\Pi(p) = \frac{1}{16\pi^2} \left[-2y^2 p^2 + 12 y^2 m_2^2 - \frac{\lambda}{2} m_1^2\right] \left[\frac{1}{\epsilon} + \ln \frac{\mu^2}{p^2}\right]. \tag{13}$$

We are concerned with the p^2 behavior and are instructed to drop the $1/\epsilon$ to enforce our counterterm subtraction prescription. Since G_0^{-1} is independent of μ, we must have that:

$$\begin{aligned} 0 &= \frac{\mu^2 d}{d\mu^2} G_0^{-1} \\ &= \frac{\mu^2 d}{d\mu^2} Z_\phi^{-1} p^2 \left[1 + \frac{2y^2}{16\pi^2} \ln \frac{\mu^2}{p^2}\right] \\ &= -Z_\phi^{-2} p^2 \frac{\mu^2 dZ_\phi}{d\mu^2} + Z_\phi^{-1} p^2 \frac{2y^2}{16\pi^2} \end{aligned} \tag{14}$$

$$\frac{\mu^2 dZ_\phi}{Z_\phi d\mu^2} = \frac{2y^2}{16\pi^2} \equiv \gamma_\phi\,. \tag{15}$$

This is the anomalous dimension. Note that $\mu^2 dZ/Zd\mu^2 = \mu dZ^{1/2}/Z^{1/2}d\mu$ which is the definition we saw in class. As expected, $\gamma_\phi > 0$.

The same approach for the fermionic self-energy identifies the coefficient on $\not{p}$ as the anomalous dimension: $\gamma_\psi = y^2/(2 * 16\pi^2)$.

1.3 Coupling

Use the Callan-Symanzik equation and the vertex corrections from the last homework, along with the anomalous dimensions from above, to determine the beta functions β_λ and β_y.

1.3.1 Solution

The bare, non-amputated correlation function has no μ dependence:

$$\frac{\mu^2 d}{d\mu^2}\langle\phi_0\phi_0\phi_0\phi_0\rangle \equiv \frac{\mu^2\, d}{d\mu^2}G_0^{(4)} = 0. \tag{16}$$

Here the superscript refers to the number of fields and the subscript says that it is the bare quantity.

Because $\phi_0 = Z_\phi^{1/2}\phi_r$, this is $G_0^{(4)} = Z^2 G_r^{(4)}$. The renormalized 4-point function is

$$Z^2 G_r^{(4)} = Z^2 (G_r)^4 G_{r,\mathrm{amp}}^{(4)} \tag{17}$$

where $(G_r)^4$ refers to 4 powers of external renormalized propagators. Since these are determined from 2-point functions, we have that $G_r = Z^{-1}G_0$ and therefore

$$\frac{\mu^2\, d}{d\mu^2} Z^{-2}(G_0)^4 G_{r,\mathrm{amp}}^{(4)} = 0\,. \tag{18}$$

The G_0 factors have no μ dependence, while the Z^{-2} factor gives $-2\gamma_\phi$, and we arrive at

$$0 = \left(-2\gamma_\phi + \frac{\mu^2\, d}{d\mu^2}\right) G_{r,\mathrm{amp}}^{(4)} \tag{19}$$

which is what we computed in the previous homework. In Euclidean space we found:

$$G_{r,\mathrm{amp}}^{(4)} = -\lambda + \frac{\frac{3}{2}\lambda^2 - 24y^4}{16\pi^2}\left(\ln\frac{\mu^2}{A} + \mathrm{const}\right) \tag{20}$$

where A is a momentum-dependent coefficient which we did not calculate. The $-\lambda$ factor multiplies $-2\gamma_\phi$, and when acted on by $\mu^2\, d/d\mu^2$ it gives β_λ. Therefore after a little algebra,

$$\beta_\lambda = \frac{\frac{3}{2}\lambda^2 + 4\lambda y^2 - 24y^4}{16\pi^2} \tag{21}$$

which is even the right answer!

The same calculation for the Yukawa interaction gives

$$\beta_y = \left(\gamma_\psi + \frac{1}{2}\gamma_\phi\right) y + \frac{y^3}{16\pi^2} = \frac{\frac{5}{2}y^3}{16\pi^2}. \tag{22}$$

Here y^3 is the term directly from the vertex correction, and the other terms are from the external line anomalous dimensions; they turn out to be larger than the vertex correction! Interestingly, the Yukawa coupling beta function is cubic in y itself. In a theory with gauge couplings as well, there will also be terms of form yg^2 with g a gauge coupling, but β_y is always proportional to at least one power of y. This ensures that $y = 0$ is preserved under renormalization group flow, which is because it is protected by the same chiral symmetry we saw in the last homework. Similarly, $\beta_g \propto g^3$. The scalar self-coupling has no such protection, as we see above, where a term of form $\beta_\lambda \propto y^4$ appears. Therefore scalar self-interactions are generically introduced as soon as scalars couple to any other degrees of freedom.

2 Renormalization Group and QED

We found in class that the beta function of QED is:

$$\frac{\mu\, de}{d\mu} = \frac{4}{3}\frac{e^3}{16\pi^2}\,. \tag{23}$$

Assume that this is true at all scales. (It's not – this is if there are only electrons!!) Use that, for $\mu = 511\text{KeV}$, that

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}\,. \tag{24}$$

Find an explicit expression for e or α as a function of μ and determine the value of μ for which e diverges. Compare this scale to the Planck scale, the scale by which gravity must become strongly coupled, which is $1.22 \times 10^{19}\text{GeV}$.

Actually, QED gets embedded into the Standard Model and becomes "hypercharge." Above the electroweak scale $\mu \sim 246$GeV, the "hyper"fine structure constant is $\alpha = 1/98$ and the beta function, featuring electrons, their heavier partners, quarks, and Higgs bosons, reads:

$$\frac{\mu\, de}{d\mu} = \frac{41}{6} \frac{e^3}{16\pi^2} . \tag{25}$$

(It's a long story to see where 41/6 comes from – let's not talk about it today.) For THIS expression, what is the scale where the coupling diverges? Is it still above the Planck scale?

2.1 Solution

We have

$$\frac{12\pi^2}{e^3} de = \frac{d\mu}{\mu} \tag{26}$$

$$\frac{6\pi^2}{e_0^2} - \frac{6\pi^2}{e^2} = \ln(\mu/\mu_0) \tag{27}$$

$$e^2 = \frac{1}{e_0^{-2} + \frac{1}{6\pi^2} \ln(\mu_0/\mu)} . \tag{28}$$

Here μ_0 is a constant of integration, as is e_0. We can choose them to be 511KeV and $e_0^2/4\pi = 1/137$. Then the coupling diverges when

$$0 = e_0^{-2} + \frac{1}{6\pi^2} \ln(\mu_0/\mu) \tag{29}$$

$$\ln(\mu/\mu_0) = \frac{6\pi^2}{e_0^2} = \frac{3\pi}{2} \frac{4\pi}{e_0^2} \tag{30}$$

$$\mu = \mu_0 \exp((3\pi/2)137) \tag{31}$$

If I put in $\mu_0 = 0.000511$GeV, I find that the divergence occurs at $\mu = 1.2 \times 10^{277}$GeV. Far above any interesting scale!

For the standard model version, $1/6\pi^2$ gets replaced by $41/48\pi^2$ and 137 gets replaced by 98, and .000511 becomes 246, and we find

$$\mu = (246) \exp((12\pi/41) \times 98) = 3 \times 10^{41} \text{ GeV}$$

which is still crazy high!

3 Renormalization group and Banks-Zacks

Write $t = \ln(\mu^2)$, so $\mu^2 dx/d\mu^2 = dx/dt$. It's simpler and more compact to study renormalization group in terms of t, and we will work in the notation where μ^2, rather than μ, is used – this is common in the modern literature.

In QCD the beta function, expressed in terms of $a = \alpha/4\pi = g^2/16\pi^2$ rather than g, can be written:

$$\frac{d\,a}{dt} = -\beta_0 a^2 - \beta_1 a^3 \,. \tag{32}$$

According to `https://arxiv.org/pdf/1701.01404.pdf`, the values of β_0 and β_1 for N_c-color, N_f-flavor QCD are:

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f \,, \tag{33}$$

$$\beta_1 = \frac{34}{3} N_c^2 - \frac{10}{3} N_c N_f - 4 \frac{N_c^2 - 1}{4N_c} N_f . \tag{34}$$

(In comparison to the reference, I used $C_A = N_c$, $T_F = 1/2$ and $C_F = (N_c^2 - 1)/(2N_c)$. Here T_F is what we called $C(F)$ in class, and C_F is what we called $C_2(F)$. These are also common notation choices, I don't know why.)

For $N_c = 3$, for what values of N_f are $\beta_0 > 0$ but $\beta_1 < 0$? In this range, the beta function has a zero at finite a value a_0. What is the value of a_0? Are there any N_c values for which this zero occurs where a_0 is small?

See if you can compute the complete t dependence of a assuming that $a(t = 0)$ lies between 0 and a_0. If you cannot, then find the behavior of $a(t)$ just in the vicinity of a_0.

3.1 Solution

The condition $\beta_0 > 0$ is $N_f > 33/2$ which is true for $N_f = 0, .., 16$. The condition $\beta_1 < 0$ is

$$0 > \frac{34}{3}3^2 - \frac{10}{3}3N_f - 4\frac{8}{12}N_f \tag{35}$$

$$0 > 102 - \frac{38}{3}N_f \tag{36}$$

$$N_f > \frac{153}{19} \tag{37}$$

which starts at $N_f = 9$. So for $N_f = 9, 10, .., 15, 16$ we have $\beta_0 > 0$ but $\beta_1 < 0$.

For the rest of the problem we just treat β_0 and β_1 as coefficients. The zero occurs where $\beta_1 a_0 = -\beta_0$ or $a_0 = -\beta_0/\beta_1$ (recall that $\beta_1 < 0$). We can rewrite

$$\frac{d\,a}{dt} = -\frac{\beta_0^2}{|\beta_1|}a^2(a_0 - a)$$

I got kinda close to solving this by rearranging,

$$\frac{da}{a^2(a_0 - a)} = -\frac{\beta_0}{a_0}dt \tag{38}$$

$$\left(\frac{1}{a_0 a^2} + \frac{1}{(a_0 - a)a_0^2} + \frac{1}{a a_0^2}\right) da = -\frac{\beta_0}{a_0}dt \tag{39}$$

$$-\frac{1}{a_0 a} + \frac{1}{a_0^2}\ln(a(a_0 - a)) = -\frac{\beta_0}{a_0}(t - t_0) \tag{40}$$

but it's not clear how to solve this for a.

To do better, let's call $a - a_0 = x$ and we will Taylor series expand about small x:

$$\frac{dx}{dt} = \frac{\beta_0^4}{\beta_1^3}x + \mathcal{O}(x^2)\,, \qquad x \simeq \exp\left(\frac{\beta_0^4}{\beta_1^3}(t - t_0)\right)\,.$$

We see that as t becomes large, x grows – the solution flows away from $a = a_0$ in the UV, towards $a = 0$ if we start with $a < a_0$ – but as t becomes very negative, $x \to 0$ – in the IR, a approaches a_0 and the theory approaches this IR interacting conformal fixed point.

4 Group theory

4.1 Fundamental representation

In carrying out some calculation, you find yourself needing to perform two group-theory calculations, in SU(3) gauge theory:

$$\text{Answer 1} = \text{Tr}\, T^A T^A T^B T^B \,, \tag{41}$$

$$\text{Answer 2} = \text{Tr}\, T^A T^B T^A T^B \,. \tag{42}$$

Here $T^A = \frac{\lambda^A}{2}$ are the fundamental-representation generators of the SU(3) Lie algebra, which are half the Gell-Mann matrices. Sums over repeated indices are implicit as usual.

First, carry out each calculation using the group-theory tricks we learned, and evaluate them using:

$$C[F] = \frac{1}{2} \tag{43}$$

$$C[A] = N_c = 3 \tag{44}$$

$$d_F = N_c = 3 \tag{45}$$

$$d_A = N_c^2 - 1 = 8 \tag{46}$$

$$C_2[F] = \frac{d_A C[F]}{d_F} = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}. \tag{47}$$

Second, carry them out using the explicit expressions for each Gell-Mann matrix, by actually conducting all of the matrix multiplications, sums, and traces involved. If I were you I would do this using Mathematica, not by hand, but you are welcome to do it by hand if you really have to.

4.1.1 Solution

Well $T^A T^A = C_2[R]$ so the first answer is

$$\text{Tr}\, T^A T^A T^B T^B = C_2[F] C_2[F]\, \text{Tr}\, 1 = d_F C_2[F]^2 = 3 \left(\frac{4}{3}\right)^2 = \frac{16}{3}$$

For the second expression, we use

$$T^A T^B = \left[T^A\,,\,T^B\right] + T^B T^A$$

The latter turns into Answer 1 because the trace is cyclic. The commutator contributes

$$\begin{aligned}
i f_{ABC}\,\mathrm{Tr}\,T^C T^A T^B &= \frac{i}{2} f_{ABC}\,\mathrm{Tr}\,T^C (T^A T^B - T^B T^A) && (48)\\
&= \frac{i^2}{2} f_{ABC} f_{ABD}\,\mathrm{Tr}\,T^A T^D && (49)\\
&= \frac{-1}{2} f_{ABC} f_{ABD} C[F]\delta_{AD} && (50)\\
&= -\frac{1}{2} d_A C[A] C[F] = -\frac{1}{2} 8 \times 3 \times \frac{1}{2} = -6 && (51)
\end{aligned}$$

Combining, we would get $16/3 - 6 = -2/3$.

I succeeded in getting Mathematica to do this calculation directly, but I won't show the transcript here. There is surely a better way than what I found.

4.2 Six representation

Oh, but wait! The particles you THOUGHT were in the fundamental representation are actually in the symmetric tensor or 6 representation! This is the representation containing $|uu\rangle$ the state with two up quarks, and all states you arrive at through raising and lowering operators from this state. We know that this rep has dimension $d_R = 6$ and Dynkin index (trace normalization) $C[R] = 5/2$, that is, $\mathrm{Tr}\,1 = 6$ and $\mathrm{Tr}\,T^A T^B = (5/2)\delta_{AB}$. Can you carry out the same calculations as before, for this rep?

For extra credit, if you are really hard-line, see if you can find somehow the actual T^A matrices for this representation, and carry out the calculation directly. I recommend that you not attempt this extra credit, but if you want to you can try.

4.2.1 Solution

First we just re-use the expressions we found, giving $C_2[R] = d_A C[R]/d_R = 8(5/2)/6 = 10/3$

$$\text{Answer 1} = d_R C_2[R]^2 = 6(10/3)^2 = \frac{200}{3} \tag{52}$$

$$\text{Answer 2} = \text{Answer 1} - \frac{1}{2} d_A C[A] C[R] = \frac{200}{3} - \frac{1}{2}(8)(3)(5/2) = \frac{200}{3} - 30 = \frac{110}{3} \tag{53}$$

For the extra credit, let's start by choosing a basis for writing the 6 representation:

$$\begin{bmatrix} rr \\ (rg+gr)/\sqrt{2} \\ gg \\ (rb+br)/\sqrt{2} \\ (gb+bg)/\sqrt{2} \\ bb \end{bmatrix} \tag{54}$$

where (r, g, b) are the three possible color combinations. Next we use that $T^1 + iT^2$ is the raising operator, which turns a g into a r. Similarly $T^1 - iT^2$ is the lowering operator, T^4 and T^5 do the same for r, b and T^6, T^7 do the same for g, b. And T^3 counts 1/2 for red and $-1/2$ for green, while T^8 counts $1/(2\sqrt{3})$ for r and g and

$-1/\sqrt{3}$ for b. Therefore, for instance, in our basis,

$$T^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{55}$$

$$\sqrt{3}T^8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \tag{56}$$

because wherever you see a state, you return the same state (diagonal entries) with a coefficient equal to the sum of charges on those entries. Now T^1 is the sum of raising and lowering operators $T^1 = (T_{r\to g} + T_{g\to r})/2$. T^2 is the same but with a relative - sign between raising and lowering and a factor of i:

$$T^1 = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{57}$$

$$T^2 = \begin{bmatrix} 0 & -i/\sqrt{2} & 0 & 0 & 0 & 0 \\ i/\sqrt{2} & 0 & -i/\sqrt{2} & 0 & 0 & 0 \\ 0 & i/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i/2 & 0 \\ 0 & 0 & 0 & i/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{58}$$

If we square any of these matrices and take the trace, we get the trace normalization or Dynkin index $C[R]$, which is indeed 5/2. If we take the product of two distinct matrices and trace, we get zero: $\mathrm{Tr}\, T^A T^B \propto \delta_{AB}$. The expressions for $T^{4,5}$ and $T^{6,7}$ are like $T^{1,2}$ but with the roles of the rows changed.

Really I should also check that these matrices satisfy the same Lie algebra as the Gell-Mann matrices (over 2), and I should try out all the expressions explicitly, but I don't have the energy and I am sure that it will work.